Category Theory Example Sheet 4

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These questions are of varying difficulty and length. Starred questions are not necessarily harder, but are extra questions. Comments, corrections and clarifications can be emailed to jg352. You can find this sheet on `www.dpmms.cam.ac.uk/~jg352/teaching.html`.

1. In a pointed category, show that $\ker(0\colon A \longrightarrow B) = 1_A$.

2. (a) Let $\mathcal{C}$ be a small category and $\mathcal{A}$ abelian. Show that the functor category $[\mathcal{C}, \mathcal{A}]$ is abelian.
   (b) Let $\mathcal{B}$ be small preadditive and $\mathcal{A}$ abelian. Prove that the full subcategory $\mathrm{Add}(\mathcal{B}, \mathcal{A}) \subset [\mathcal{B}, \mathcal{A}]$ of additive functors $\mathcal{B} \longrightarrow \mathcal{A}$ is abelian.
   (c) Show that for a (unitary) ring R, the category R-Mod of (left) R-modules is isomorphic to $\mathrm{Add}(R, \mathsf{AbGp})$.

3. **Additive Yoneda Lemma**
   (a) If $\mathcal{A}$ is a preadditive category and A is an object in $\mathcal{A}$, prove that the "representable functor" $\mathcal{A}(A, -)\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$ is additive.
   (b) Given an object A in a preadditive $\mathcal{A}$ and $F\colon \mathcal{A} \longrightarrow \mathsf{AbGp}$, prove that there exists an isomorphism of abelian groups
   $$\theta_{A,F}\colon \mathrm{Nat}(\mathcal{A}(A, -), F) \cong F(A)$$
   which is natural in A and F.

4. A category $\mathcal{A}$ is called *semi-additive* if it is enriched in the category of monoids. In this question you will prove that in certain cases a semi-additive structure exists (and then it is unique, see proof for additive structures in lectures).
   (a)* Let X be a set equipped with a distinguished element 0, and two binary operations $+$ and $*$ both of which have 0 as a (two-sided) identity element, and which satisfy the 'middle interchange law'
   $$(x + y) * (z + w) = (x * z) + (y * w) \ .$$
   Show that $+$ and $*$ coincide and that they are (it is?) associative and commutative (i.e., X is a commutative monoid). [This is a well-known piece of pure algebra, which I've included here in case you haven't seen it before.]
   (b) Now let $\mathcal{A}$ be a locally small pointed category with finite products and coproducts, where the product of any two objects coincides with their coproduct (more precisely, the functors $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ sending (A, B) to $A \times B$ and to $A + B$ are naturally isomorphic). By considering the distinguished element $0\colon A \longrightarrow 0 \longrightarrow B$ of $\mathcal{A}(A, B)$ and the two binary operations on this set sending (f, g) to the composites
   $$A \xrightarrow{(1,1)} A \times A \cong A + A \xrightarrow{(f,g)} B \qquad \text{and} \qquad A \xrightarrow{(f,g)} B \times B \cong B + B \xrightarrow{(1,1)} B$$
   respectively, show that $\mathcal{A}$ is semi-additive.

5. A *pseudo-mono* is a morphism $f\colon A \longrightarrow B$ such that $fg = 0$ implies $g = 0$.
   (a) Show that if $\mathcal{A}$ is preadditive, then any pseudo-mono in $\mathcal{A}$ is a mono.
   (b) Let $\mathcal{C}$ be pointed with kernels and cokernels, such that every mono in $\mathcal{C}$ is normal. Show that every morphism in $\mathcal{C}$ factors as a pseudo-epi followed by a mono. [Given $f\colon A \longrightarrow B$, let $k = \ker \operatorname{coker}(f)$, and prove that the factorisation g of f over k is a pseudo-epi.]

6.* The following 'addition-free' definition of an abelian category is often found in textbooks: $\mathcal{A}$ is abelian if it has a zero object, binary products, binary coproducts, kernels and cokernels, every monomorphism in $\mathcal{A}$ is a kernel and every epimorphism is a cokernel. Show that this definition is equivalent to the one given in lectures, along the following lines:

(a) Show that $\mathcal{A}$ has pullbacks of pairs $(f\colon A \longrightarrow C, g\colon B \longrightarrow C)$ one of which is monic [hint: consider the kernel of qf, where q is the cokernel of g], and deduce that $\mathcal{A}$ has equalizers (and hence all finite limits).

(b) Dually, $\mathcal{A}$ has all finite colimits. Now show that any pseudo-mono in $\mathcal{A}$ is monic. [If f is pseudo-monic and $fx = fy$, let q be a coequalizer of x and y: note that q is epic, and hence a cokernel of some morphism z, but f factors through q and hence z factors through $0 \longrightarrow A$.]

(c) Given two objects A and B, consider the morphism $f\colon A + B \longrightarrow A \times B$ with matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Show that f is both monic and epic, and hence an isomorphism. [Hint: first show that $\iota_2\colon B \longrightarrow A + B$ is the kernel of $(1, 0)\colon A + B \longrightarrow A$.] Deduce that $\mathcal{A}$ has a semi-additive structure.

(d) Finally, obtain the additive inverse of a morphism $f\colon A \longrightarrow B$ by considering the (multiplicative!) inverse of the morphism $A \oplus B \longrightarrow A \oplus B$ with matrix $\left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)$.

7. (a) Show that in the category $\mathsf{AbGp}_{t.f.}$ of torsion-free abelian groups, not every monomorphism is a kernel and not every epimorphism is a cokernel. [Warning: epimorphisms in this category do not have to be surjective.]

(b) Let $\mathcal{C}$ be the category of finitely-generated abelian groups having no elements of order 4 (though they may have elements of order 2), and homomorphisms between them. Show that every epimorphism in $\mathcal{C}$ is (surjective, and hence) a cokernel, but not every monomorphism in $\mathcal{C}$ is a kernel.

(c) Let $\mathcal{A}$ be a preadditive category with kernels and cokernels, in which every epimorphism is a cokernel but not every monomorphism is a kernel. Show that normal monomorphisms (equivalently, regular monomorphisms) in $\mathcal{A}$ must fail to be closed under composition. [Given a non-normal monomorphism f, factor it as kg where k is the kernel of the cokernel of f; then let l be the kernel of the cokernel of g, and show that kl is not a normal monomorphism.]

8. Let $\mathcal{A}$ be abelian. Consider

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \qquad \text{and} \qquad A \xrightarrow{\left(\begin{smallmatrix} f \\ -g \end{smallmatrix}\right)} B \oplus C \xrightarrow{(h,k)} D$$

Prove

(a) $(h, k)\begin{pmatrix} f \\ -g \end{pmatrix} = 0$ iff the square commutes.

(b) $\begin{pmatrix} f \\ -g \end{pmatrix} = \ker(h, k)$ iff the square is a pullback.

(c) $(h, k) = \operatorname{coker}\begin{pmatrix} f \\ -g \end{pmatrix}$ iff the square is a pushout.

9. Use the Nine Lemma to prove the Noether's Third Isomorphism Theorem: In an abelian category $\mathcal{A}$, consider subobjects $A \rightarrowtail B \rightarrowtail C$. Then B/A is a subobject of C/A and $(C/A)/(B/A) \cong C/B$.