Groups Example Sheet 1

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Please send comments and corrections to jg352.

The questions are not necessarily in order of difficulty. Particularly questions 12 and 13 are just questions on a later topic, so do make sure you look at them! You should attempt all questions on the sheet in any case and hand in your best solutions. Starred or "exploration" questions should not be done to the detriment of later sheets or other subjects.

1. Let G be any group. Show that the identity e is the unique solution of the equation $a^2 = a$.

2. Let H_1 and H_2 be two subgroups of the group G.

   Show that the intersection $H_1 \cap H_2$ is a subgroup of G.

   Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H_i contains the other.

3. Let G be a finite group.

   (a) Let $a \in G$. Show that there is a positive integer n such that $a^n = e$, the identity element. (The least such positive n is the *order* of a.)

   (b) Show that there exists a positive integer n such that $a^n = e$ for all $a \in G$. (The least such positive n is the *exponent* of G.)

4. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element a of G has finite order.

5. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} : |z| = 1\}$. Show that S is a group, and deduce that for some $n \in \mathbb{N}$, S is the set of n-th roots of unity; that is, $S = \{\exp(2k\pi i/n) : k = 0, \dots, n-1\}$.

6. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

7. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that if G is finite, the order of G is a power of 2. [Consider a minimal generating set. A minimal generating set is a set which generates G but no proper subset of which generates G.]

8. Let G be a group of even order. Show that G contains an element of order two.

9. Let G be a finite group and f a homomorphism from G to H. Let $a \in G$. Show that the order of $f(a)$ is finite and divides the order of a.

10. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

11. How many homomorphisms $D_{2n} \longrightarrow C_n$ are there? How many isomorphisms $C_n \longrightarrow C_n$?

12. Write these permutations as products of disjoint cycles and compute their order and sign:

    (a) $(12)(1234)(12)$;

    (b) $(123)(235)(345)(45)$.

13. What is the largest possible order of an element in S_5? And in S_9?
    Show that every element in S_{10} of order 14 is odd.

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14. * Which groups contain a (non-zero) even number of elements of order 2?

15. * Let G be the set of integers modulo 2^n with operation

$$x * y = 4xy + x(-1)^y + y(-1)^x \pmod{2^n}$$

    Show that G is a cyclic group.

16. * *Exploration question* Is there any operation on the natural numbers $\mathbb{N}$ which makes it into a group? If yes, how many (non-isomorphic) such groups can you find?

    [If you want hints, you can look on the Moodle site in the forum.]