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MA1114 Linear Algebra - Workbook

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Contents

How to use the workbook

You will be set some of these questions to hand in to your Feedback Session leaders to receive feedback, and some questions you will do in Feedback Sessions. Other questions are left for you to do for your own practice or revision. Some questions have a link to Numbas: this way, you can get immediate feedback.
If a question says "Define ...." or "[bookwork]", this means that you can check your own answer with your lecture notes. It is still important that you try to do these questions without looking at your lecture notes first, especially when you come to revision.
In some sections, there may be more questions than I am going to set for handing in and feedback sessions. This is on purpose so that you also have some questions for revision.

Chapter 0. A-level recap

It is quite possible that you will find these recap questions easy. If you don't find them easy, then please work on the concepts until you are comfortable. We will need to be comfortable with these so we can concentrate on the new material.

0.1. If the point $(x, 3x - 4)$ is a general point on a given line,

a) is the point $(4, 3 \cdot 4 - 4)$ on the line?

b) Is the point $(2n, 6n - 4)$ on this line?

c) Is the point $(4u, 12u - 16)$ on this line?

d) Is the point $(u + v, 3(u + v) - 4)$ on this line?

0.2. Consider the pattern $(x_1, x_2, 2x_1 - x_2)$.

a) Is $(-1, 3, -5)$ of this pattern?

b) Is $(1, 1, 3)$ of this pattern?

c) Is $(a, b, 2a - b)$ of this pattern?

d) Is $(a + c, b - d, 2a - b + 2c + d)$ of this pattern?

e) Is $(lu_1 + kv_1, lu_2 + kv_2, 2lu_1 - lu_2 + 2kv_1 - kv_2)$ of this pattern?

0.3. Translate the following in and out of sigma sum notation.

a) Write out $\sum_{i=1}^{4} a_i$ in full.

b) Write $b_1^1 + b_2^2 + b_3^3 + \cdots + b_n^n$ with a summation sign $\sum$.

c) Write out $\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij}^{i+j}$ in full.

Chapter 1. Vectors and Matrices

Vectors, lines and planes

Introductory exercises

1.1. Calculate the following:

$\diamond\ 0 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \diamond\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \diamond\ 3 \cdot \begin{pmatrix} \sqrt{3} \\ \pi \end{pmatrix} + 6 \cdot \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \pi \end{pmatrix}$

Or do question 1.1 in Numbas, or a randomised version of question 1.1 in Numbas.

Essential practice

1.2. With a friend, give each other vectors in $\mathbb{R}^3$ to add.

Or add vectors of random sizes in Numbas.

1.3. Define the expression **linear combination of vectors**.

1.4. Consider the lines $\left\{ \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$ and $\left\{ \begin{pmatrix} 8 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$.

- ◇ Verify that $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ are points on the first line.
- ◇ Show that $k \begin{pmatrix} -3 \\ -6 \end{pmatrix}$ is on the first line for any $k \in \mathbb{R}$.
- ◇ Show that $\begin{pmatrix} -3 \\ -6 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ is on the first line.
- ◇ Determine two different points on the second line.
- ◇ Show that if you add those two points, the sum is not on the second line.

1.5. Define what it means for two vectors to be **parallel**.

1.6. Determine which of the following are planes.

You can do this question on planes in Numbas.

For each plane, also say whether it goes through the origin or not. (You can use the tick boxes given.) CAREFUL: while we said that if the plane (or line) goes through the origin, we don't use the fixed first vector (the one that does not have a parameter in front of it), the following planes may not be written in the most efficient way, so you have to check explicitely whether they go through the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ or not.

a) $\left\{ \lambda \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$

b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$

c) $\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ -6 \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$

d) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$

e) $\left\{ \begin{pmatrix} -1 \\ 3 \\ \frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -\frac{3}{2} \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$

Stretch yourself

1.7. If you have two lines in $\mathbb{R}^2$, consider their intersection. What are the possibilities for the number of points in which they intersect? What about if the lines are in $\mathbb{R}^3$? Does it change or stay the same?

1.8. Given two planes in $\mathbb{R}^3$, what are the possibilities for their intersection?

Vector space $\mathbb{R}^n$

Introductory exercises

1.9. Make up any vector in $\mathbb{R}^5$ and then write down its negative. Or find the negative of a random vector in Numbas.

Essential practice

1.10. For any vector $v \in \mathbb{R}^n$, what is $0 \cdot v$? What is $1 \cdot v$, and $(-1) \cdot v$? You can do question 1.10 in Numbas.

1.11. You can do this question on general linear combinations in Numbas.

a) Can you write any vector $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$ as a linear combination of the vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$?

b) Can you write any vector $\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{smallmatrix}\right) \in \mathbb{R}^4$ as a linear combination of the vectors $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right)$? Can you do it with just the first three of these vectors?

1.12. Verify the associativity of scalar multiplication in $\mathbb{R}^3$: given $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ and $\lambda,\ \mu \in \mathbb{R}$, show that $\lambda \cdot (\mu v) = (\lambda \cdot \mu) v$.

1.13. Let $\left\{\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \lambda \in \mathbb{R}\right\}$, with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$, be any line through 0.

- ◇ Show that this line is a subspace of $\mathbb{R}^2$.
- ◇ Show that a line which does not go through zero is not a subspace of $\mathbb{R}^2$.

1.14. Show that the following are subspaces of $\mathbb{R}^4$:

◇ $\left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R}\right\}$ ◇ $\left\{\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} \middle| x_1, x_2, x_4 \in \mathbb{R}\right\}$ ◇ $\left\{\begin{pmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_2, x_3, x_4 \in \mathbb{R}\right\}$

1.15. Look at the vector space rules given in "Properties of vector addition and scalar multiplication" (Prop. 1.21).

a) If you take vectors with rational entries, rather than real number entries, do the rules still hold? (You can think about $\mathbb{Q}^2$ instead of $\mathbb{Q}^n$.)

b) Does it make a difference whether you allow the scalars λ, μ to be any real number or restrict them also to rationals?

c) If either of those examples fail, say which of the rules fail.

Stretch yourself

1.16. Show that the intersection of two subspaces of $\mathbb{R}^n$ is again a subspace.

Matrices

Introductory exercises

1.17. Determine the sizes of the following matrices:

◇ $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ ◇ $\begin{pmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}$ ◇ $\begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}$ ◇ Find the matrix sizes in Numbas.

1.18. Decide whether the following statements are true or false:

a) A 6×3 matrix has 6 rows.

b) A 4×7 matrix has 7 rows.

c) A $m \times n$ matrix has m columns.

True/false about matrix sizes in Numbas.

Essential practice

1.19. Define the notion of a **matrix** and its **entries**.

1.20. If it is possible, add the following matrices.

- $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2 \\ 8 & 2 \end{pmatrix} + \begin{pmatrix} 9 & 1 & 0 \\ 3 & 8 & 4 \\ 9 & 2 & 2 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2 & 9 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 2 \\ 9 & 8 & 1 \end{pmatrix}$
- $4 \cdot \begin{pmatrix} 9 & 1 & 8 \\ 3 & 7 & 3 \\ 8 & 1 & 7 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 & 9 & 1 \\ 8 & 2 & 7 \\ 3 & 6 & 4 \end{pmatrix}$

Use Numbas for "if possible, add matrices".

1.21. a) Define the **trace** of a square matrix.

b) Calculate the trace of the following matrices:

- $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix}$
- $\begin{pmatrix} 9 & 1 & 0 \\ 3 & 8 & 4 \\ 9 & 2 & 2 \end{pmatrix}$
- $\begin{pmatrix} 2 & 8 & 8 & 1 \\ 7 & 2 & 2 & 1 \\ 10 & -4 & 2 & -2 \\ 19 & -100 & 15 & 29 \end{pmatrix}$

Calculate these traces in Numbas. Calculate a random trace in Numbas.

1.22. Let

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}, \quad F = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Compute all possible sums of these matrices, and the scalar multiple of all matrices with $\lambda = 3$. Compute matrix sums and scalar multiplications in Numbas.

1.23. Find the 4×4 matrix $A = (a_{ij})_{4\times 4}$ whose entries satisfy the stated condition:

- $a_{ij} = i + j$
- $a_{ij} = i^{j-1}$

Find matrix from entry formula in Numbas. Find random size matrix from entry formula in Numbas.

1.24. Let $A,\ B,\ C$ be $m \times n$-matrices, 0 the $m \times n$ zero matrix, and $\lambda,\ \mu \in \mathbb{R}$. Of the following statements, prove as many as you need to, so you can then write a guide to an (imaginary) fellow student how to go about proving these kinds of statements about matrices.

a) $\lambda(B+C) = \lambda B + \lambda C$

b) $A + (-A) = 0$

c) $A + (B+C) = (A+B) + C$

d) $(\lambda + \mu)C = \lambda C + \mu C$

e) $\lambda(\mu C) = (\lambda\mu)C$

f) $A + 0 = 0 + A = A$

g) "How to prove these statements" guide

What do these statements tell you about $m \times n$ matrices?

1.25. Let $A, B \in \mathcal{M}_{n,n}$. Prove that

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)\,.$$

Stretch yourself

1.26. Prove that matrices form a vector space, i.e. that they satisfy all the axioms we gave for the vector space $\mathbb{R}^n$.

(Tip: use ideas from Question 1.24 to make this about understanding and theory, and not about repetitious tedium ;-))

Matrix multiplication

Introductory exercises

1.27. Calculate the following:

- $\begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 3 & 2 \\ 9 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 1 & 9 & 0 \\ 8 & 0 & 1 \\ 0 & 2 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
- $\begin{pmatrix} 1 & 9 \\ 3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Matrix times vector in Numbas. Randomised matrix times vector in Numbas.

1.28. Let

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix},\ D = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix},\ E = \begin{pmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$$

Determine whether the product exists, and give the size of the result (without computing the product): $AE,\ EA,\ BE,\ CE,\ DE,\ ED$. Determine product size in Numbas.

Essential practice

1.29. Define the notion of

a) the **product** of a matrix with a vector.

b) a **linear combination** of matrices.

c) the **product** of two matrices.

d) the **transpose** of a matrix.

1.30. Let

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}, \qquad v = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}.$$

a) Compute Au, Av, Bu, Bv.

b) Write down the matrix transformation of A in the form $T_A \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ (i.e. determine m and n), and for a general vector x, determine $T_A(x)$.

Q 1.30 in Numbas. Randomised version of Q 1.30 in Numbas.

1.31. Let

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}, D = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}, E = \begin{pmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$$

a) Which matrix products out of AB, BA, AC, CA, AD, DA exist? (These are different ones than in Question 1.28.) Compute AB and BA if they exist, and at least one more of the ones that exist.

b) Compute the transpose of matrices A, B, C, D, E. Compute, if possible, $A^T B^T$ and $B^T A^T$, and compare to AB and BA (if they exist).

c) Compute $A(BC)$, $(AB)C$, CC^T.

d) What do you notice in all of this? Link anything you notice back to the theory from lectures.

Do Q1.31 a)-c) in Numbas.

Optional: Check any combination of matrix mult from these matrices in Numbas.

1.32. For "Matrix multiplication is linear" (Prop. 1.67):

a) Write the calculation out in long hand, without summation symbols, for two 2×2 matrices.

b) Write out the proof for the second equation, with summation symbols. Which aspects of this proof are different to the one of the first equation? Which are the same?

1.33. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

a) Work out $(AB)C$ and $A(BC)$ for these matrices. Show that they are the same.

b) Compare your calculations with the ones in the proof of "Associativity of matrix multiplication" (Prop. 1.59). Which step in the proof corresponds to which step in your calculation?

c) Think about how this generalises to bigger matrix sizes. Why do you think we wrote the proof down with summation notation?

d) Now show that for the above matrices, $\lambda(AB) = A(\lambda B) = (\lambda A)B$. Compare to the calculation from the first part: do you see a difference in complexity between this and $(AB)C = A(BC)$, or do they have the same complexity?

1.34. Decide whether the follow statements are true or false.

For each false statement, give a counterexample.

a) If A, B and C are square matrices of the same size such that $AC = BC$, then $A = B$.

b) If $AB + BA$ is defined, then A and B are square matrices of the same size.

c) If B has a column of zeros, then so does AB if this product is defined.

d) If B has a column of zeros, then so does BA if this product is defined.

e) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

f) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$.

1.35. For which k is

$$\begin{pmatrix} k & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix} = 0\,?$$

Matrix multiplication with indeterminate in Numbas.

1.36. Let

$$A = \begin{pmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{pmatrix}$$

a) Compute A^2 and B^2.

b) Express the first column vector of A^2 as a linear combination of the column vectors of A.

c) Express the second column vector of B^2 as a linear combination of the column vectors of B.

Matrix multiplication via linear combinations of columns in Numbas.

1.37. a) Construct two 2×2 matrices A and B which are not zero but such that $AB = 0$.

b) Now construct two more pairs of such matrices, which are a little different. Say why it your examples are different.

c) Discuss what is different and what is the same in your examples. For example, can you make one of the matrices have no zero entries at all? Can you make both matrices have no zero entries at all?

d) What is the largest number of zeros you can put in without either of the matrices being the zero matrix?

Matrix product giving zero in Numbas.

1.38. Let

$$A = \begin{pmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{pmatrix}$$

The matrix A gives the matrix transformation $T_A \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $T_A(v) = Av$. Write down the matrix transformations which correspond to AB, BA, A^2, B^2 in the same way as this. (You do not have to calculate the matrix products for this.) How do they relate to the matrix transformations for A and B?

1.39. Consider the matrix transformation $T_A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

a) Compute $T_A(x)$ for

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

b) How would you describe this function geometrically?

c) Verify that $T_A(x + y) = T_A(x) + T_A(y)$ and $T_A(\lambda x) = \lambda T_A(x)$ for $x, \; y \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Stretch yourself

1.40. a) Use the Σ notation to show that if A, B are square matrices of the same size then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

b) Prove that if AB and BA are both defined, then AB and BA are square matrices.

c) Show that if $A \in \mathcal{M}_{m,n}$ and $A(BA)$ is defined, then $B \in \mathcal{M}_{n,m}$.

1.41. Politics: In the matrix

$$P = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{pmatrix}$$

each entry P_{ij}, $i \neq j$, represents the proportion of the voting population that changes from party i to party j, and P_{ii} represents the proportion that remains loyal to party i from one election to the next. Find and interpret the product of P with itself.

Chapter 2. Linear systems

Linear equations

Introductory exercises

2.1. What is a linear system? Give an example with two and three variables. Give a counter-example.

2.2. Determine whether the following matrices are diagonal, upper or lower triangular, or neither. Diagonal and triangular matrices in Numbas.

a) $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix}$ c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{pmatrix}$ d) $\begin{pmatrix} 9 & 1 & 0 \\ 3 & 8 & 0 \\ 9 & 2 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 7 & 2 & 0 & 0 \\ 10 & -4 & 2 & 0 \\ 19 & -100 & 15 & 29 \end{pmatrix}$

Essential practice

2.3. Consider the two lines given by the equations

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 - 2x_2 &= b\,, \end{aligned}$$

with $b \in \mathbb{R}$. For which choices of b, if any, are the lines parallel? For which choices of b, if any, do the lines intersect in a single point? For which choices of b, if any, are the lines equal? Justify your answers.

Draw the lines for some sample values of b.

2.4. a) Define the notion of a consistent linear system.

b) Define the notion of an inconsistent linear system.

c) Define the notion of a general solution of a linear system.

2.5. Find the matrix A and vectors b, x with which you can write the following linear system as a matrix equation. Translate system into matrix in Numbas.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -3 \\ 2x_1 + x_2 &= 0 \\ -3x_2 + 4x_3 &= 1 \\ x_1 + x_3 &= 5 \end{aligned}$$

2.6. Write the line $x + y = 0$ in parametric form.

Write this line in Numbas. Write randomised line in Numbas.

Stretch yourself

2.7. In lectures we saw that two lines can be

parallel (no intersection), intersect in one point, or coincide (infinite intersection points).

Consider three planes in $\mathbb{R}^3$. In how many different ways can they intersect, and how many vectors are in the common intersection of all three in each of the cases?

2.8. Give an example of a line in $\mathbb{R}^5$.

Write your example both in form of a linear system and in parametric form.

Elementary row operations and echelon forms

Introductory exercises

2.9. Use the matrices from Question 2.2 and calculate the following elementary row operations (you can choose a matrix for each):

- ◇ Swap the first and the last row.
- ◇ Multiply the second row by -100.
- ◇ Add 3 times the first row to the second row.

Calculate elementary row operations in Numbas. You will be given a random one of these matrices for each.

Essential practice

2.10. Are the following operations elementary row operations? Are they the result of several elementary row operations? Or can they not be achieved by elementary row operations? Justify your answers.

a) Add 0 times the first row to the second row.

b) Multiply the last row by 0.

c) Cycle round three rows: first to second, second to third, third to first.

2.11. a) Define when a matrix is in echelon form.

b) Define when a matrix is in reduced echelon form.

2.12. Determine whether the matrix is in row echelon form (but not reduced), reduced row echelon form, or neither. Check echelon form in Numbas.

$$\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0\\0&2\end{pmatrix} \quad \begin{pmatrix}1&0&-\pi\\0&1&-17\\0&0&0\end{pmatrix} \quad \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\begin{pmatrix}1&5\\-3&1\end{pmatrix} \quad \begin{pmatrix}1&0&-17\\0&1&1\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

2.13. These matrices are in row echelon form. Bring them into *reduced* row echelon form. In Numbas: a) RREF, b) RREF, c) RREF, d) RREF

$$\text{a)}\ \begin{pmatrix}1&8&-2\\0&1&-2\\0&0&1\end{pmatrix} \qquad \text{b)}\ \left(\begin{array}{ccc|c}1&3&2&1\\0&0&1&1\end{array}\right) \qquad \text{c)}\ \begin{pmatrix}0&1&2&4\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} \qquad \text{d)}\ \left(\begin{array}{cccc|c}1&-1&2&3&1\\0&1&1&2&-1\\0&0&0&1&0\end{array}\right)$$

2.14. Determine whether the following systems are consistent, and if yes, give the general form of the solution. (This should probably be in the next section, but if I move it now, all the Numbas question numbers will go wrong.)

Consistent and solution to RREFs in Numbas.

$$\text{a)}\ \left(\begin{array}{ccc|c}1&0&0&0\\0&1&2&0\\0&0&0&1\end{array}\right) \qquad \text{b)}\ \left(\begin{array}{cccc|c}1&0&0&0&1\\0&1&2&0&2\\0&0&0&1&-1\end{array}\right) \qquad \text{c)}\ \begin{pmatrix}1&0&2\\0&1&-1\\0&0&0\end{pmatrix} \qquad \text{d)}\ \left(\begin{array}{ccc|c}1&-5&1&4\\0&0&0&0\end{array}\right)$$

2.15. Determine whether the statement is true or false. For false statements, give a counter-example. For true statments, give a brief justification.

a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

b) If an elementary row operation is applied to a matrix that is in row echelon form, then the resulting matrix will still be in row echelon form.

c) Every matrix has a unique row echelon form.

Stretch yourself

2.16. Working on a general 2×2 matrix $A = \begin{pmatrix}a&b\\c&d\end{pmatrix}$,

find matrices E_1, E_2 and E_3 such that

- ◇ $E_1 A$ is the matrix obtained from A by multiplying row I by λ;
- ◇ $E_2 A$ is the matrix obtained from A by adding λ times row I to row II;
- ◇ $E_3 A$ is the matrix obtained from A by swapping rows I and II.

How would you have to change E_1 so that multiplying by E_1 corresponds to multiplying row II by λ (instead of row I)? And how would you have to change E_2 so that multiplying by E_2 corresponds to adding λ times row II to row I?

What happens if you calculate AE instead of EA for any of these elementary matrices?

Elementary matrices in Numbas.

Gauss algorithm

Introductory exercises

2.17. On the augmented matrix

$$\left(\begin{array}{cccc|c} 5 & -10 & 20 & 15 & -5 \\ 3 & 27 & 23 & -2 & 63 \\ 6 & 3 & 8 & 13 & 87 \end{array}\right)$$

perform the following elementary row operations:

- ◇ Multiply row I by $\frac{1}{5}$.
- ◇ Add -3 times row I to row II.
- ◇ Add -6 times row I to row III. Now you should have $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in the first column.
- ◇ Multiply row II by $\frac{1}{33}$.
- ◇ Add -15 times row II to row III. Now your second column should be $\begin{pmatrix} * \\ 1 \\ 0 \end{pmatrix}$.
- ◇ Multiply row III by $-\frac{1}{21}$. Now your matrix should be in row echelon form.
- ◇ Add $-\frac{1}{3}$ times row III to row II.
- ◇ Add -4 times row III to row I. Now the third column should be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
- ◇ Add 2 times row II to row I. Now the second column should be $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: the matrix should be in reduced row echelon form.

2.18. The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

is in reduced row echelon form. Give the solution of the corresponding linear system in parametric form. Solution from RREF in Numbas.

Essential practice

2.19. Solve the following two systems by using the Gauss-Jordan algorithm. Use matrix notation. Write the solution set in parametric form.

a) Solve system a) in Numbas

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

b) Solve system b) in Numbas

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= -1 \\ 2x_1 + x_2 - 2x_3 - 2x_4 &= -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 &= 1 \\ 3x_1 \qquad\qquad\quad - 3x_4 &= -3 \end{aligned}$$

2.20. Are the following linear systems consistent or inconsistent? If possible, find a solution to the linear system. Justify your answer. Use matrix notation. (You can use the Gaussian Elimination Calculator, but write down all your steps.)

a)

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

b)

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + 3x_2 &= 9 \\ x_1 + x_2 &= 2 \end{aligned}$$

c)

$$\begin{aligned} x_1 + x_2 + x_3 &= -6 \\ x_1 + 2x_2 + 3x_3 &= -10 \end{aligned}$$

2.21. Determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions. Give all solutions for those a for which the system is consistent. Use the Gauss-Jordan algorithm and matrix notation.

$$\begin{array}{cccl} x & +2y & -z & = 2 \\ 2x & -2y & +3z & = 1 \\ x & +2y & -(a^2-3)z & = a \end{array}$$

Stretch yourself

2.22. Find the coefficients a, b, c and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

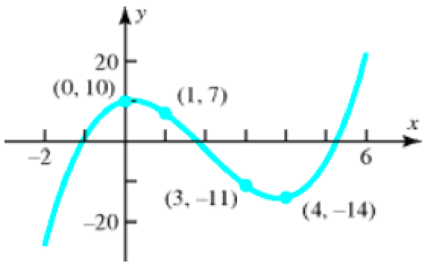

Figure Ex-37

2.23. Solve the following systems of *nonlinear* equations for x, y and z.

a)

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

b)

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

You may ask me for a hint if you can't get started.

Theory about solutions

Introductory exercises

2.24. Determine whether the statement is true or false.

- A linear system can have exactly two solutions.
- If v_0 is a solution to a given inhomogeneous sytem, then adding any solution v of the corresponding homogeneous system to v_0 gives another solution of the inhomogeneous system.
- It is possible to have an inconsistent homogeneous linear system.

Linear systems true/false in Numbas.

Essential practice

2.25. Determine whether the statement is true or false.

- ◇ A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n-r$ free parameters in the general solution.
- ◇ All leading 1's in a matrix in row echelon form must occur in different columns.
- ◇ If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.
- ◇ If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.
- ◇ If a linear system has more unknowns than equations, then it must have infinitely many solutions.

2.26. a) Prove that if $ad - bc \neq 0$ then the reduced echelon form of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hint: consider the case $a = 0$ and $a \neq 0$ separately.

b) Use the result of (a) to show that if $ad - bc \neq 0$ then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

2.27. Prove that if the homogeneous linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

has a unique solution, then

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

has a unique solution for any choice of b_1, b_2, b_3.

(We will revisit this question and do it again differently in the next chapter.)

2.28. Decide whether the following statements are true or false:

a) Every matrix is row-equivalent to a matrix in echelon form.

b) If the echelon form of the augmented matrix of a system of linear equations contains the row $\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, then the original system is inconsistent.

c) A homogeneous system of four linear equations in six variables has infinitely many solutions.

d) A homogeneous system of four linear equations in four variables is always consistent.

e) Multiplying a row of a matrix by any constant is one of the elementary row operations.

Chapter 3. Inverse matrices and determinants

Inverse matrices

Introductory exercises

3.1. Show that $A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

are inverses of each other.

3.2. Find the inverse of $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Introductory inverse in Numbas. Randomised introductory inverse.

Essential practice

3.3. a) Define the notion of an invertible matrix.

b) Define the notion of the inverse of an invertible matrix.

3.4. Find the inverses of the following matrices:

a) $A = \begin{pmatrix} 2 & -3 \\ 4 & 4 \end{pmatrix}$ b) $B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ c) $C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ d) $D = \frac{1}{4}A$ e) $E = A^T$.

Can you describe what B and C do geometrically?

Hint: consider what they do to the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Inverses of 2×2 in Numbas.

3.5. Look at Proposition 3.7 (Properties of inverse).

a) What is the main technique that gets used in the proofs of all three parts?

b) Write down which previous results and concepts these proofs use or link to.

c) Look at/work through Example 3.8 to see the difference between $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. How do we use this different order in the proof in Proposition 3.7?

d) Example 3.8 gives an example where $A^{-1}B^{-1} \neq B^{-1}A^{-1}$. In fact, this will be the case for most matrices that you might choose. Can you come up with an example of 2×2 matrices where in fact $A^{-1}B^{-1} = B^{-1}A^{-1}$? Tip: consider diagonal matrices.

e) Can you change the example in Example 3.8, leaving $A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ but changing $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ slightly, so that $A^{-1}B^{-1} = B^{-1}A^{-1}$?

3.6. Simplify $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$.

Simplify in Numbas.

3.7. a) Use "socks and shoes" to show that if A is invertible, then for any natural number k, A^k is invertible with $(A^k)^{-1} = (A^{-1})^k$.

b) Prove that if A is invertible, then for any $\lambda \neq 0$, so is λA, with $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.

3.8. Show that if

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{22} & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & & \dots & & 0 & a_{nn} \end{pmatrix}$$

with $a_{11}a_{22} \cdots a_{nn} \neq 0$, then A is invertible. Find the inverse of A.

3.9. Decide whether the follow statements are true or false:

a) Two matrices A and B are inverse to each other if and only if $AB = BA = 0$.

b) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

c) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

d) The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

e) If A is an invertible matrix, then so is A^T.

f) The sum of two invertible matrices of the same size must be invertible.

3.10. a) Explain in your own words why a matrix with a row of zeros cannot have an inverse.

b) Show that if $A,\ B$ and $A + B \in \mathcal{M}_{n,n}$ are invertible, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I_n\,.$$

Conclude that in general $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

3.11. Look at Proposition 3.13 (Inverse matrix transformation).

a) Explain in your own words what an inverse function does.

b) Explain why this proof shows that $T_A^{-1} = T_{A^{-1}}$.

c) Why do we need to show the two conditions in the proof? Why not just one of them?

d) Can you find two functions f and g (they don't have to be matrix transformations, just any functions) where $f \circ g$ is the identity function but $g \circ f$ is not?

3.12. A square matrix $A \in \mathcal{M}_{n,n}$ is said to be **idempotent** if $A^2 = A$.

a) Show that if A is idempotent, then so is $I_n - A$.

b) Show that if A is idempotent, then $2A - I_n$ is invertible and is its own inverse.

(Idem means self, and potent refers to the fact that we're taking powers.)

Stretch yourself

3.13. Let A be an $n \times n$ matrix. Show that if $A^k = 0$ for k positive integer, then $I_n - A$ is invertible and

$$(I_n - A)^{-1} = I_n + A + A^2 + \ldots + A^{k-1}\,.$$

If $A^k = 0$ for some natural number k, we call the matrix **nilpotent**. (Nil for zero, and potent for some power of it is zero.)

Elementary matrices and inverse algorithm

Introductory exercises

3.14. Decide whether the matrix is an elementary matrix:

a) $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$ b) $\begin{pmatrix} -5 & 1 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$ f) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}$

Decide if elementary matrix in Numbas.

3.15. Calculate the following:

a) $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Product with elementary matrix in Numbas.

Essential practice

3.16. Define the notion of an elementary matrix.

3.17. a) In each part, an elementary matrix E and a matrix A are given. Write down the row operation corresponding to E and show that the product EA is the same as the result of applying the row operation to A.

(i) $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{pmatrix}$

(ii) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{pmatrix}$

b) Given

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{pmatrix},$$

find elementary matrices E_i such that $E_1A = B$, $E_2B = A$, $E_3A = C$, $E_4C = A$.

Find elementary matrices in Numbas. (Part b) only)

3.18. Determine the product AB by inspection.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{pmatrix}$$

Product by inspection in Numbas.

3.19. Find the elementary 3×3 matrix E which is given by the elementary row operation which adds 2 times the second row to the third.

Compute EA where $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$.

Elementary matrix on general 3×2 in Numbas.

3.20. Write down the inverses of the following elementary matrices:

a) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Inverse elementary matrices in Numbas.

3.21. Use the inversion algorithm to find the inverses of:

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & 0 \\ -1 & 2 & 6 \end{pmatrix}$$

3.22. Use the inversion algorithm to decide whether the matrix is invertible, and if so to find the inverse of the given matrix:

$$A = \begin{pmatrix} 1 & 2 & -2 & 7 \\ -3 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 2 & 1 & 11 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$$

Do you notice a connection between B and C?

3.23. Use the inversion algorithm

to decide whether A is invertible, and if so, to find its inverse:

$$A = \begin{pmatrix} \frac{1}{2} & 2 & -2 & 7 & 1 \\ -\frac{1}{3} & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & 1 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

3.24. a) Prove that a triangular matrix is invertible if and only if its diagonal entries are all non-zero.

b) Without computing the inverse, determine whether AB is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

Without computation, decide whether A and B are invertible. (Justify your answer).

Stretch yourself

3.25. Let E be an elementary matrix obtained by an elementary row operation which adds a multiple of one row to another. Prove (in general!) that EA is the matrix which is obtained by applying the same elementary row operation to A.

Hint: Define E_{ij} for the different possibilities of i, j.

Inverses and Linear Systems

Introductory exercises

3.26. The linear system

$$3x_1 + 5x_2 = -1 \qquad \text{or} \qquad Ax = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$
$$x_1 + 2x_2 = 5$$

has $A^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$. Calculate the unique solution of the system.

Invertible linear system intro in Numbas.

Essential practice

3.27. Referring to Proposition 3.14, explain in your own words why a linear system $Ax = b$ with invertible matrix A has a unique solution for any b. Can it happen that $Ax = b$ has a unique solution for one particular b but not another b'?

3.28. Solve the following linear systems by using the inverse matrix.

a)
$$3x_1 - 2x_2 = -1$$
$$4x_1 + 5x_2 = 3$$

b)
$$6x_1 + x_2 = 0$$
$$4x_1 - 3x_2 = -2$$

Invertible linear systems in Numbas.

3.29. Use the **definition of an invertible matrix** to decide whether $A = \begin{pmatrix} 2 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{R}$, is invertible. If so, find its inverse.

(I.e. don't use the formula for 2×2 matrices, and don't use the inverse algorithm.)

3.30. Use the **definition of an invertible matrix** to decide whether

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is invertible. If so, find its inverse.

(I.e. don't use the garden fence formula for 3×3 matrices, and don't use the inverse algorithm.)

3.31. Make a list or "concept diagram" of all results and concepts that go into the proof of Corollary 3.27: if A, $B \in \mathcal{M}_{n,n}$ and $AB = I_n$ then A is invertible, and $A^{-1} = B$.

Stretch yourself

3.32. We have proved that if A is invertible, then $Ax = b$ has a unique solution (for any b). Is it true that if a given linear system $Ax = b$ has a unique solution, then A must be an invertible matrix? (As always, justify your answer.)

Determinants

Introductory exercises

3.33. Calculate/determine the determinants of the following elementary matrices:

a) $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ b) $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ c) $E_3 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Determinant of elementary matrices in Numbas.

3.34. Verify the garden fence rule

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

by expanding along the first row.

Essential practice

3.35. Calculate the determinant of the following matrices:

a) $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$ b) $B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & -3 \\ -1 & 3 & -3 & 3 \end{pmatrix}$

Intro dets in Numbas. Intro dets randomised.

3.36. For the following matrices, find $\widehat{A}_{ij}$ for as many pairs i, j as you need so that you can be sure you will be able to do it for any matrix and any i, j. Also calculate $\det A$ in at least two different ways: e.g. expand along row 1 and column 3. Verify that you get the same answer from both calculations. Also verify that $\det A^T = \det A$. That means calculate both and see that you get the same answer.

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ -6 & -7 & 1 \\ 2 & -2 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

Part a) A hat in Numbas. Part b) A hat in Numbas.

3.37. Define the notion of a determinant.

3.38. Evaluate the determinants of the following matrices using row reductions.

$$A = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 0 & -1 \\ 2 & -1 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

3.39. Construct examples of a matrix which has determinant 0

a) but no zero entries;

b) but no repeated rows or columns or zero rows or columns;

c) but no zero entries and no repeated rows or columns.

3.40. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, evaluate the following determinants.

a) $\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$

b) $\begin{vmatrix} 3a & b+2c & c \\ 3d & e+2f & f \\ 3g & h+2i & i \end{vmatrix}$

Dets and row/column operations in Numbas.

3.41. Prove that the determinant of a lower triangular matrix is the product of its diagonal entries.

3.42. Look at the proof of Theorem 3.55:

a square matrix A is invertible if and only if $\det A \neq 0$.

a) List the properties of determinant that are needed in this proof. Where in the proof is which property used?

b) What is the link of this proof with the Gauss algorithm?

c) Why do elementary matrices have non-zero determinant?

d) Write down at which points in the proof we use the fact that A (and R) is a square matrix.

3.43. a) What is the determinant of an $n \times n$ matrix in which every entry is 1?

b) What is the maximum number of 0s that a 3×3 matrix can have without its determinant being 0? Explain your reasoning.

3.44. Let $a_1, \ldots, a_n$ and b_k be columns of $n \times n$ matrices.

Prove that the determinant is linear in columns (i.e. prove Proposition 3.21(ii)):

$$\det\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_k+b_k & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix} = \det\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_k & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix} + \det\begin{pmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & b_k & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{pmatrix}$$

Explain why the same holds for rows.

3.45. Let A and B be $n \times n$ matrices, and $\lambda \in \mathbb{R}$.

a) Prove that $\det(\lambda A) = \lambda^n \det A$.

b) Prove that $\det(AB) = \det(BA)$.

3.46. Look at the proof of $\det(AB) = \det(A)\det(B)$.

a) Why can we treat elementary matrices differently when considering the determinant of a product of matrices?

b) Do A and B play really different roles in this proof? What happens if B is singular instead of A?

Stretch yourself

3.47. Prove that swapping two columns introduces a minus sign to the determinant, i.e. Proposition 3.21(iii), in the following way using Induction:

Let A be an $n \times n$ matrix, and let B be the matrix obtained from A by swapping columns k and l.

$$A = \begin{pmatrix} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_k & \cdots & a_l & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \end{pmatrix} \qquad B = \begin{pmatrix} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_l & \cdots & a_k & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \end{pmatrix}$$

Without loss of generality, assume $k \neq 1$ and $l \neq 1$. We want to prove $\det B = -\det A$.

- ◇ We know this is true for 2×2 matrices. (Reprove it by direct calculation, or look it up in your notes.)
- ◇ So assume that it is true for $(n-1) \times (n-1)$ matrices.
- ◇ For the induction step, expand $\det B$ in column 1, and use the induction hypothesis on all the $\det \widehat{B}_{i1}$. Justify carefully why we can use the induction hypothesis there, and how $\widehat{B}_{i1}$ and $\widehat{A}_{i1}$ are related.

3.48. Find the determinant of the $n \times n$ matrix which has a 1 in every off-diagonal entry and a λ in each diagonal entry. For which λ is the determinant 0? (If you can't see how to do $n \times n$ at first, try 3×3 and 4×4 to get an idea.)

Chapter 4. Vector spaces

Vector spaces and subspaces

Introductory exercises

4.1. Show that the set $\mathcal{M}_{m,n}$ of $m \times n$ matrices forms a real vector space, using the following steps:

- ◇ Identify the set V whose elements are the vectors (given above). So in this case, the $m \times n$ matrices are our "vectors", i.e. elements of the vector space.
- ◇ Identify the operations of addition and scalar multiplication. (I.e. how are these defined in your example of V?)
- ◇ Verify that these operations satisfy VA0 and SM0. We then say that V is **closed under addition and scalar multiplication**.
- ◇ Verify that there is a zero "vector" (i.e. VA1) and that there are negative "vectors" (i.e. VA2).
- ◇ Verify all the other axioms.

Essential practice

4.2. Define the notion of a vector space.

4.3. a) Show that the set $\mathbb{R}^\infty$ of infinite sequences forms a vector space, with addition and scalar multiplication as defined in lectures.

b) Show that the set P of all real polynomials forms a vector space.

4.4. Let V be the set of real numbers together with addition given by $u + v = uv$ for $u,\ v \in V$, and scalar multiplication $ku = u^k$ for $u \in V$, $k \in \mathbb{R}$.

Is V a vector space?

Note that it is not quite the same as our example in lectures: there we said we only use positive real numbers. Will it all work out the same if you use all real numbers? Or will the negative numbers break something?

4.5. Determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail. Check vector space in Numbas.

- ◇ The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication.
- ◇ The set of all 2×2 diagonal matrices.
- ◇ The set of pairs of real numbers with operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (kx, y)$.

4.6. Let V be the set of ordered pairs

$$V = \{u = (u_1,\ u_2) \mid u_1,\ u_2 \in \mathbb{R}\}\,,$$

and consider the following addition and scalar multiplication operations for $u,\ v \in V$ and $\lambda \in \mathbb{R}$:

$$u + v = (u_1 + v_1, u_2 + v_2), \qquad \lambda u = (10, \lambda u_2)\,.$$

a) Compute $u + v$ and λu for $u = (-1, 2),\ v = (3, 4)$.

b) Explain why V is closed under addition and scalar multiplication.

c) Which vector space axioms hold?

d) Show that V is not a vector space.

4.7. Refer to Proposition 4.6 in your notes - zeros and negatives in vector spaces.

Let V be a real vector space, and $v \in V$, $\lambda \in \mathbb{R}$.

- ⋄ Prove statement (iii): For any $\lambda \in \mathbb{R}$, $\lambda 0 = 0$: any multiple of the zero vector is still the zero vector. Indicate at each step which vector space axiom you are using.
- ⋄ Go through the proofs of statements (ii) and (iv) and indicate whenever you see a 0 whether it is the real number 0 or the zero vector in V.
- ⋄ Compare the proof of statement (v) (negatives are unique) with the proof of Prop. 3.6 (inverse matrices are unique). Explain how the operations and results used in each of these two proofs correspond to each other, i.e. explain why "this is the same proof".

4.8. Define the notion of a subspace of a vector space.

4.9. Let V be a vector space, and $U,\ W,\ W_1,\ \ldots,\ W_k$ subspaces of V.

a) Show that the intersection $U \cap W$ is a subspace of V.

b) Conclude that the intersection $W_1 \cap \cdots \cap W_k$ is a subspace of V.

c) Explain why if $U \subseteq W$, then $U \cup W$ is a subspace of V.

d) Show that if the union $U \cup W$ is a subspace of V, then either $U \subseteq W$ or $W \subseteq U$.

4.10. Determine which of the following are subspaces of $\mathbb{R}^3$:

a) All vectors of the form $(a, 0, 0)$.

b) All vectors of the form $(a, 1, 1)$.

c) All vectors of the form (a, b, c), where $b = a + c$.

d) All vectors of the form (a, b, c), where $b = a + c + 1$.

e) All vectors of the form $(a, b, 0)$.

4.11. Let $A = (a_{ij})_{n\times n}$ be an $n\times n$ matrix. Determine whether A is symmetric or anti-symmetric (or neither). Check symmetric/anti-symmetric in Numbas.

a) $a_{ij} = i^2 + j^2$ b) $a_{ij} = i^2 - j^2$ c) $a_{ij} = 2i + 2j$ d) $a_{ij} = 2i^2 + 2j^3$

4.12. a) Prove that a skew-symmetric matrix $A \in \mathcal{M}_{n,n}$ has vanishing diagonal entries.

(Vanishing entries means the entries are zero. This term is sometimes used by mathematicians, so I'm using it here so you've seen it and know what it means if you come across it.)

b) Prove that the set of skew-symmetric matrices is a subspace of $\mathcal{M}_{n,n}$.

4.13. Which of the following sets are subspaces of the polynomial space P_n?

a) P_{n-1}

b) $XP_{n-1} = \{\text{polynomials with zero constant term}\} = \{p = a_nX^n + \cdots + a_2X^2 + a_1X\}$

c) $\{\text{polynomials of exactly degree } n\}$

d) $\{\text{polynomials which have 1 as a root}\} = \{(X-1)q \mid q \in P_{n-1}\}$

e) $\{\text{polynomials which have constant term 1}\} = \{p = a_nX^n + \cdots + a_2X^2 + a_1X + 1\}$

4.14. For each of the following subspaces, determine whether the sum $U + W$ is a direct sum, and whether the sum gives the whole vector space V (i.e. whether $U + W = V$).

Direct sums and whole space in Numbas.

a) $V = \mathbb{R}^2$, $U = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$ and $W = \left\{ \begin{pmatrix} -y \\ y \end{pmatrix} \middle| \, y \in \mathbb{R} \right\}$.

b) $V = \mathbb{R}^3$, $U = \left\{ \begin{pmatrix} x \\ x \\ -x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$ and $W = \left\{ \begin{pmatrix} y \\ z \\ -z \end{pmatrix} \middle| \, y, \ z \in \mathbb{R} \right\}$.

c) $V = \mathbb{R}^3$, $U = \left\{ \begin{pmatrix} x \\ y \\ -y \end{pmatrix} \middle| \, x, \ y \in \mathbb{R} \right\}$ and $W = \left\{ \begin{pmatrix} 0 \\ z \\ -z \end{pmatrix} \middle| \, z \in \mathbb{R} \right\}$.

d) $V = P_2$ (polynomials of degree at most 2), $U = \{p = a_2X^2 + a_1X \mid a_1, \ a_2 \in \mathbb{R}\}$ and $W = P_1 = \{q = b_1X + b_0 \mid b_1, \ b_0 \in \mathbb{R}\}$.

e) $V = P_2$ (polynomials of degree at most 2), $U = \{p = a_2X^2 \mid a_2 \in \mathbb{R}\}$ and $W = P_0 = \{q = b_0 \mid b_0 \in \mathbb{R}\}$.

4.15. Let P be the space of all polynomials, and P_n the space of polynomials of degree at most n.

a) Prove that $P_n \leq P$ (i.e. P_n is a subspace of P).

b) Prove that there is a nested sequence of subspaces

$$P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_n \leq \cdots$$

of P.

4.16. Look at Prop. 4.22, Symmetric and anti-symmetric matrices.

a) Work through the proof using the self-explanation technique.

b) Get someone to give you a random square matrix (say 3×3 or 4×4), or make one up yourself. Then use the proof to write this particular matrix as the sum of a symmetric matrix and an anti-symmetric matrix.

Stretch yourself

4.17. (This uses some Analysis content. For some items, you may need to wait till you have covered it in Analysis.)

a) Show that the set of all real functions

$$F = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$$

forms a vector space, with addition and scalar multiplication defined point-wise.

Show that all the following are subspaces of F:

b) The set of all differentiable functions

$$D = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid \text{the derivative of } f \text{ exists}\} .$$

c) The set of all continuous functions

$$C = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ is a continuous function}\} .$$

d) The set of all continuously differentiable functions

$$C^1 = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid \text{the derivative } f' \text{ of } f \text{ exists and } f' \text{ is a continuous function}\} .$$

e) The set of all twice continuously differentiable functions

$$C^2 = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid \text{both } f' \text{ and } f'' \text{ exists and are continuous functions}\} .$$

f) The set of all infinitely differentiable functions

$$C^\infty = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ can be differentiated infinitely many times}\} .$$

g) Show that these subspaces are nested: they sit inside each other, for example $C^2 \subseteq C^1$. Determine the order of the containment.

Column space and nullspace

Introductory exercises

4.18. Show that $\begin{pmatrix}1\\2\end{pmatrix}$ is in the span of $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

Intro span in Numbas. Intro span randomised.

4.19. Is the column space of $A = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$ all of $\mathbb{R}^2$?

Intro column space in Numbas. Intro column space randomised.

4.20. Determine the null space of $A = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$.

Intro null space in Numbas. Intro null space randomised.

Essential practice

4.21. a) Define the notion of the span of a set S.

b) When do we say that S **spans** V?

c) Give an example of a spanning set for $\mathbb{R}^3$, and for P_2.

4.22. For each example, determine whether w is in the span of v_1, v_2. If yes, write w as a linear combination of v_1 and v_2 (i.e. determine the coefficients for the linear combination). You can solve this either by inspection, or using the method described in the notes under "Finding a linear combination".

Check if the vector is in the span in Numbas.

a) $v_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}-3\\-6\end{pmatrix}$, $w = \begin{pmatrix}-4\\8\end{pmatrix}$

b) $v_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}-3\\-6\end{pmatrix}$, $w = \begin{pmatrix}4\\8\end{pmatrix}$

c) $v_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $w = \begin{pmatrix}4\\5\\5\end{pmatrix}$

d) $v_1 = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $w = \begin{pmatrix}-4\\-4\\-5\end{pmatrix}$

e) $v_1 = \begin{pmatrix}-1\\1\\9\\3\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\-1\\-3\\5\end{pmatrix}$, $w = \begin{pmatrix}9\\-5\\-33\\1\end{pmatrix}$

4.23. Consider Prop. 4.24, Span gives subspace.

a) Make yourself an example set S with say 4 actual example vectors from $\mathbb{R}^n$, where $n = 4$ or 5. Work through the steps of the proof on your example S.

b) Make yourself an example set S with 3 polynomials of degree at most 5. Work through the steps of the proof on this example S.

c) In each case, if the span of S does not give the whole vector space, try to find another subspace which contains all elements of S but is bigger than the span of S. If possible, find one that is not all of V. (This is not always possible, depending on your example. Can you work out when it is possible and when not?)

4.24. Determine whether the given vectors span all of V or not. In each case where they do, write out the linear combination that gives a general vector in terms of the given vectors, for example $\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+y}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{x-y}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Span all of V or not in Numbas.

a) $V = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b) $V = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

c) $V = \mathbb{R}^3$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

d) $V = \mathbb{R}^3$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

4.25. Let $S = \{v_1, v_2, \ldots, v_k\}$ and $S' = \{w_1, w_2, \ldots, w_m\}$ be two non-empty subsets of a vector space V.

a) Suppose that all vectors in S can be written as linear combinations of vectors in S':

$$\begin{aligned} v_1 &= \lambda_{1,1} w_1 + \lambda_{1,2} w_2 + \cdots + \lambda_{1,m} w_m \\ v_2 &= \lambda_{2,1} w_1 + \lambda_{2,2} w_2 + \cdots + \lambda_{2,m} w_m \\ &\vdots \\ v_k &= \lambda_{k,1} w_1 + \lambda_{k,2} w_2 + \cdots + \lambda_{k,m} w_m \end{aligned}$$

Show that $\mathrm{span}(S) \subseteq \mathrm{span}(S')$: the span of S is contained in the span of S'. (You do this by proving that a general element in $\mathrm{span}(S)$ is also in $\mathrm{span}(S')$.)

b) Explain why what you showed in a) shows:

$$S \subseteq \mathrm{span}(S') \quad \Rightarrow \quad \mathrm{span}(S) \leq \mathrm{span}(S')$$

c) Show that if every element of S' can be written as a linear combination of elements in S, then $\text{span}(S') \leq \text{span}(S)$.

d) Explain why $S \subseteq \text{span}(S)$: the set S is contained in its span.

e) Explain why if $\text{span}(S) \leq \text{span}(S')$, then every element of S can be written as a linear combination of elements of S'.

f) Put everything together to prove Proposition 4.8 (Equal spans):
Given two non-empty subsets S and S' of a vector space V, then $\text{span}(S) = \text{span}(S')$ if and only if every vector in S is a linear combination of vectors in S', and every vector in S' is a linear combination of vectors in S.

4.26. Define what the **column space** of a matrix is.

4.27. In each case, determine whether the given vector b is in the column space of the given matrix A. If yes, write b as a linear combination of the columns of A. (This linear combination may or may not be unique. You can check if it is correct by adding up the linear combination yourself.)

Check if b in column space in Numbas.

a) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 8 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & -4 & -2 \\ 2 & 2 & 6 \\ -3 & 1 & -5 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 8 \\ -8 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -4 & -2 \\ 2 & 2 & 6 \\ -3 & 1 & -5 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 5 \\ -8 \end{pmatrix}$

4.28. Define the **null space** of a matrix.

4.29. In each case, determine the null space of the given matrix.

a) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{pmatrix}$ b) $B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & 3 \end{pmatrix}$

Null space part a) in Numbas Null space part b) in Numbas

c) $C = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ d) $D = \begin{pmatrix} 1 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$ e) $E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

Null space c) Numbas Null space d) Numbas Null space e) Numbas

Compare your solutions with Questions 2.19 and 2.20 from Chapter 2 Workbook, and think about how the theory connects this question with those.

4.30. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 4 \end{pmatrix}$$

Construct matrix with given null space in Numbas.

Stretch yourself

4.31. In your own words, explain the connection of column space and null space to the theory of solutions of homogeneous and/or inhomogeneous linear systems.

Chapter 5. Linear Independence and Bases

Linearly independent sets

Introductory exercises

5.1. Determine if the following sets are linearly independent or not.

a) $\{0\}$ in $\mathbb{R}^4$. b) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ in $\mathbb{R}^3$. c) $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$ in $\mathbb{R}^2$.

d) $\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-4\\4\\0\end{pmatrix}\right\}$ in $\mathbb{R}^3$. e) $\{p_1 = x,\ p_2 = x^2,\ p_3 = 0\}$ in P_2.

Determine introductory linear independence on Numbas.

Essential practice

5.2. Explain why the following are linearly dependent sets of vectors. (Solve this problem by inspection.)

a) $\left\{v_1 = \begin{pmatrix}-1\\2\\4\end{pmatrix}, v_2 = \begin{pmatrix}5\\-10\\-20\end{pmatrix}\right\}$

b) $\left\{v_1 = \begin{pmatrix}3\\-1\end{pmatrix}, v_2 = \begin{pmatrix}4\\5\end{pmatrix}, v_3 = \begin{pmatrix}-4\\7\end{pmatrix}\right\}$

c) $\{p_1 = 3 - 2x + x^2,\ p_2 = 6 - 4x + 2x^2\}$

5.3. Give your own examples of linearly dependent sets (not ones seen in lectures or another WBQ),

a) one which is an example for Corollary 5.5 ("easy to spot dependent sets") (i);

b) one which is an example for Corollary 5.5 ("easy to spot dependent sets") (iii);

c) one which is an example for Proposition 5.7 ("Too many vectors").

5.4. Determine whether the vectors lie in a plane.

a) $v_1 = \begin{pmatrix}2\\-2\\0\end{pmatrix}, v_2 = \begin{pmatrix}6\\1\\4\end{pmatrix}, v_3 = \begin{pmatrix}2\\0\\-4\end{pmatrix}$

b) $v_1 = \begin{pmatrix}-6\\7\\2\end{pmatrix}, v_2 = \begin{pmatrix}3\\2\\4\end{pmatrix}, v_3 = \begin{pmatrix}4\\-1\\2\end{pmatrix}$

Vectors in plane in Numbas.

5.5. Show that the three vectors

$$v_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ -1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 6 \\ 0 \\ 5 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 4 \\ -7 \\ 1 \\ 3 \end{pmatrix}$$

form a linearly dependent set in $\mathbb{R}^4$.

Express each of the vectors as a linear combination of the others.

Show dependency in Numbas.

5.6. Determine which of the following sets are linearly independent.

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -10 \\ -2 \\ -8 \end{pmatrix} \right\}$ in $\mathbb{R}^3$

b) $\left\{ \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ -1 \\ 1 \end{smallmatrix}\right) \right\}$ in $\mathbb{R}^4$

c) $\left\{ \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ 1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ -1 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ -1 \\ -3 \end{smallmatrix}\right) \right\}$ in $\mathbb{R}^4$

5.7. Determine all real values λ for which the following vectors form a linearly dependent set in $\mathbb{R}^3$.

$$v_1 = \begin{pmatrix} \lambda \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \; v_2 = \begin{pmatrix} -\frac{1}{2} \\ \lambda \\ -\frac{1}{2} \end{pmatrix}, \; v_3 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \lambda \end{pmatrix}$$

5.8. Let $v, \; w \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$. Show that the following two statements are equivalent.

(i) $v \neq 0$ and $w \neq rv$ for all $r \in \mathbb{R}$.

(ii) $\lambda v + \mu w = 0 \Rightarrow \lambda = \mu = 0$.

5.9. Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$.

a) Put together previous results from the course to show that the n vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ are linearly independent if and only if $\det(A) \neq 0$.

b) Formulate a condition for the vectors $v_1, v_2, \ldots, v_n$ to be linearly independent which involves the null space of A.

5.10. a) Prove that if S_1 is a nonempty subset of the finite set S_2, and S_1 is linearly dependent, then so is S_2.

b) Prove that a nonempty subset T of a finite set S of linearly independent vectors is linearly independent.

Stretch yourself

5.11. The set S consists of six vectors in $\mathbb{R}^4$:

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Find two different subsets X and Y of S which are linearly independent, each of which yields a linearly dependent subset of S whenever any extra vector from S is added to the set.

Bases Part I

Introductory exercises

5.12. Show that the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},, \; \dots, \; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

(i.e. the *standard basis*) really form a basis of $\mathbb{R}^n$.

Essential practice

5.13. This question is meant to help your visualisation and intuition, and also help you to make links between different ideas.

For the whole question, let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a) Draw an x, y-coordinate system, and draw in the vectors v_1, v_2, v_3. From looking at them, do you think they are linearly independent or not?

b) Are vectors v_1 and v_2 linearly independent? Are vectors v_1 and v_3 linearly independent? Are vectors v_2 and v_3 linearly independent?

c) Solve the linear system $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$. Do you get any non-zero solutions for $\lambda_1, \lambda_2, \lambda_3$?

d) Compare your answers to the first three parts. Think about what this tells you. Discuss with friends what you think it tells you.

e) Now look at vectors $w_1 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, $w_2 = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$, $w_3 = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$. Are they linearly independent or not? How do you prove it? What about w_1, w_3 and $w_4 = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$?

5.14. Prove that the following are bases of the given vector space.

a) $\{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ of $\mathbb{R}^2$.

b) $\{w_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ of $\mathbb{R}^2$.

c) $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

d) $\left\{ v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 2 \\ 9 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

e) $\left\{ E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathcal{M}_{2,2}$.

f) Let E_{ij} be the $m \times n$ matrix with zeros everywhere, except a 1 in the i, jth entry. Show that $E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, E_{31}, \ldots, E_{m1}, \ldots, E_{mn}$ form a basis of $\mathcal{M}_{m,n}$.

5.15. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\} \quad S_2 = \left\{ \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} \right\} \quad S_3 = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ 1 \end{pmatrix} \right\}$$

Check basis or not in Numbas.

5.16. Give an example of a basis of $\mathbb{R}^4$ which is not the standard basis. Check your basis in Numbas.

We will learn more about bases in Chapter 6, so we'll do some more questions on them then.

Coordinate vectors

Introductory exercises

5.17. Find the coordinate vector of v with respect to basis B:

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \qquad B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Intro coordinate vector in Numbas.

5.18. Find the coordinate vector for the polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

with respect to the standard basis $1, x, x^2, \ldots, x^n$ of P_n. Poly coordinate vector in Numbas.

Essential practice

5.19. Find the coordinate vector of v relative to the basis B:

$$v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\}.$$

Coordinate vector in Numbas. Coordinate vector randomised, same basis. Coordinate vector randomised, similar basis. Coordinate vector randomised, harder basis.

5.20. Define the notion of a coordinate vector with respect to a basis.

5.21. Let $v, w \in V$, $\lambda, \mu \in \mathbb{R}$ and $v_1, v_2, \ldots, v_n$ be a basis B of V. Show that

$$[\lambda v + \mu w]_B = \lambda [v]_B + \mu [w]_B.$$

In words: the coordinate vector of a linear combination is the linear combination of the coordinate vectors.

5.22. Prove that $v_1, \ldots, v_n$ is a basis of V if and only if every $v \in V$ can be expressed uniquely as a linear combination of the v_i.

[One direction was done in lectures, in "uniqueness of basis representation". You should use the self-explanation technique on that part to really understand it. Also prove the other direction.]

Chapter 6. Bases and dimension

Dimension

Introductory exercises

6.1. Show that the following dimensions are correct by writing down a basis for the given vector space with the correct number of vectors: a) $\dim \mathbb{R}^n = n$, b) $\dim P_n = n+1$, c) $\dim \mathcal{M}_{m,n} = mn$.

6.2. Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Intro dimension of span in Numbas.

Essential practice

6.3. a) Define the notion of a basis.

b) Define the notion of the dimension of a vector space.

6.4. Find the dimension of the vector spaces spanned by the following sets of vectors (c.f. Question 5.15).

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right\} \quad S_2 = \left\{ \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} \right\} \quad S_3 = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ 1 \end{pmatrix} \right\}$$

I.e. if $W_1 = \mathrm{Span}(S_1)$ is the subspace of $\mathbb{R}^3$ spanned by the vectors in S_1, find $\dim W_1$, and the same for S_2 and S_3.

Dimension of span in Numbas.

6.5. Find the dimension of the following vector spaces, explaining your argument:

a) the space of all diagonal $n \times n$ matrices.

b) the space of all symmetric $n \times n$ matrices.

c) the space of all upper triangular $n \times n$ matrices.

d) the space of all anti-symmetric $n \times n$ matrices (those that satisfy $A^T = -A$).

In each case, write down a basis for the case $n = 3$. (You need not prove that it is a basis if it is sufficiently obvious, e.g. there are lots of 0 entries.)

6.6. Find a basis for the nullspaces of A and B.

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis for nullspace in Numbas.

6.7. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Show that the column space of A has dimension 2 if and only if one or more of the determinants

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

is non-zero.

6.8. a) Define the notion of cartesian product of two vector spaces.

b) Explain why the cartesian product $V \times W$ of two vector spaces is again a vector space. (You can either prove it in detail, or explain how you would go about proving it and why it will work out.)

c) Explain in your own words how we build a basis for a cartesian product out of bases for the individual spaces, and why it works. (You may refer to Proposition 6.9 but you should summarise/explain in your own words, not copy the proof. You can use the self-explanation technique to help you.)

d) Explain what dimension formula this new basis for the cartesian product gives us, and give two examples of spaces V and W and the dimensions of V, W and $V \times W$ in both your examples.

Stretch yourself

6.9. We will look at the relation between dimension of null space and dimension of column space in this question. (We'll prove such a relation in a more general setting later in the course.)

Let

$$A = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & 5 & 8 & 3 \\ 1 & -2 & 4 & -3 \\ 5 & -2 & 20 & -7 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \\ 2 & 0 & 8 \end{pmatrix}.$$

a) Find the reduced row echelon form for both A and B.

b) Make a link between the properties of the reduced row echelon forms (e.g. number of leading 1s, number of stuff columns, number of zero rows, or others which might help you) and the dimension of the null space of the matrix.

c) The reduced REF of A has two "stuff columns" (columns 3 and 4) and two "pivot columns", i.e. columns with leading 1s (columns 1 and 2). The reduced REF of B has one stuff column (column 2), and two pivot columns.

   Explain why you can write the original of each stuff column (i.e. the original column 3 or 4 of A, or the original column 2 of B) as a linear combination of the original pivot columns (i.e. the original columns 1 and 2 of A, or original columns 1 and 3 of B).

d) Explain on matrix B why you may *not* be able to write one of the original pivot columns as a linear combination of the original stuff columns.

e) Explain why deleting columns 3 and 4 out of your calcluation to find the reduced REF of A shows that the first two columns of A are linearly independent. Explain why deleting column 2 from B in your calculations shows that the first and third column of B are linearly independent.

f) From the above, show that the original pivot columns of the matrix form a basis for the column space of the matrix.

g) Deduce that the following dimension equation is true for matrices A and B:

$$\text{dimension of column space } + \text{ dimension of null space} = n$$

   where the matrix has size $n \times n$.

h) Now explain why these arguments will hold for any $n \times n$ matrix: taking the columns that turn into pivot columns gives a basis for the column space, and the above dimension formula holds.

Bases Part II

In Chapter 6 we have learnt more about bases, for example

- ◇ any two bases in the same vector space have the same size;
- ◇ Check One Get One Free for Bases.

So we'll look at some more questions about bases here.

Essential practice

6.10. Explain why the following sets of vectors are not bases for the indicated vector spaces.

a) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ for $\mathbb{R}^2$;

b) $v_1 = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$ for $\mathbb{R}^3$;

c) $p_1 = 1 + x + x^2,\ p_2 = x - 1$ for P_2.

Explain why not basis in Numbas.

6.11. Show that the following matrices form a basis for $\mathcal{M}_{2,2}$.

$$A_1 = \begin{pmatrix} 3 & 6 \\ 3 & -6 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & -8 \\ -12 & -4 \end{pmatrix},\ A_4 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

6.12. Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Determine whether the following sets form a basis of $\mathbb{R}^n$. The answer may depend on n.

a) $v_1 = e_1,\ v_2 = e_2 - e_1,\ \dots,\ v_k = e_k - e_{k-1},\ \dots,\ v_n = e_n - e_{n-1}$.

b) $v_1 = e_1 - e_n,\ v_2 = e_2 - e_1,\ \dots,\ v_k = e_k - e_{k-1},\ \dots,\ v_n = e_n - e_{n-1}$.

c) $v_1 = e_1 + e_2,\ v_2 = e_2 + e_3,\ \dots,\ v_k = e_k + e_{k+1},\ \dots,\ v_n = e_n + e_1$.

d) $v_1 = e_1 - e_n,\ v_2 = e_2 + e_{n-1},\ v_3 = e_3 - e_{n-2},\ \dots,\ v_n = e_n + (-1)^n e_1$.

6.13. Let P_n be the vector space of polynomials of degree at most n.

a) Prove that $1,\ x,\ x^2, \dots,\ x^n$ form a basis of P_n.

b) Prove that $1,\ x + 1,\ (x + 1)^2, \dots,\ (x + 1)^n$ form a basis of P_n.

c) Prove that $1,\ x + 1,\ x^2 + x + 1, \dots\ x^n + x^{n-1} + \dots + x + 1$ is a basis of P_n.

Stretch yourself

6.14. Let $x,\ y,\ z$ be real numbers.

a) Show that

$$\det\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix} = (x-y)(y-z)(z-x).$$

b) Show that $v_1 = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 \\ z \\ z^2 \end{pmatrix}$ form a basis of $\mathbb{R}^3$ if and only if $x,\ y,\ z$ are all distinct.

The determinant in a) is called a **Van der Monde determinant**, and you can do it for an $n \times n$ matrix:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & \vdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

But it is much harder to calculate for large matrices. The answer is a similar product of all differences of these variables $x_1, \ldots, x_n$.

Plus/Minus Theorem

Introductory exercises

6.15. Extend $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ to a basis of $\mathbb{R}^3$.

Extend to basis intro on Numbas.

6.16. Reduce $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ to a basis for $\mathbb{R}^3$.

Reduce to basis intro on Numbas.

Essential practice

6.17. a) The vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ are linearly independent. Extend $\{v_1, v_2\}$ to a basis of $\mathbb{R}^3$. Also think: can you keep your extra vector(s) as simple as possible?

Extend to basis parts a), b) in Numbas.

b) Extend the following vectors to a basis of $\mathbb{R}^4$.

$$v_1 = \begin{pmatrix} -1 \\ 4 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -8 \\ -4 \\ 5 \end{pmatrix}$$

c) Extend the following polynomials to a basis of P_3.

$$p_1 = 3 + 2x - x^2, \quad p_2 = -1 + 3x + 2x^2 - x^3$$

6.18. Reduce the following sets to bases of the vector space given. You may take for granted that the given set spans the given vector space.

a) $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ v_4 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \ v_5 = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \right\}$, in $\mathbb{R}^3$.

b) $\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ v_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \ v_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \ v_6 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} \right\}$, in $\mathbb{R}^5$.

Hint: which of the first 5 vectors do you have to add together to get the last one?

Reduce to basis in Numbas.

6.19. Find a basis for the subspace of P_2 spanned by the given vectors. If the subspace is not all of P_2, extend the linearly independent set you find to a basis of all of P_2.

a) $p_1 = -1 + x - 2x^2$, $p_2 = 3 + 3x + 6x^2$, $p_3 = 9$.

b) $p_1 = 1 + x$, $p_2 = x^2$, $p_3 = -2 + 2x^2$, $p_4 = -3x$.

c) $p_1 = 1 + x - 3x^2$, $p_2 = 2 + 2x - 6x^2$, $p_3 = 3 + 3x - 9x^2$.

[Hint: If you prefer vectors, translate these polynomials into coordinate vectors for the basis $B = \{1, x, x^2\}$.]

6.20. Using the result "equal spans" (Prop. 4.27), explain why performing elementary column operations on a matrix does not change the column space.

6.21. Look at the Plus/Minus Theorem and its proof.

a) Make an example of a linearly independent set S in a vector space V of your choice, and a vector v which is not in the span of S (as in the condition of part (i) of the theorem), and another vector w which *is* in the span of S. (You may use one of Examples 6.12 to give you ideas, if you need to.)

b) Now take your two examples $S \cup \{v\}$ and $S \cup \{w\}$ and follow the proof of (i) through on these examples. How does it work on your specific $S \cup \{v\}$? What goes wrong when you follow it through on $S \cup \{w\}$, which does not satisfy the conditions?

c) Repeat for part (ii) of the theorem: make two examples, one set S which spans V and say the final vector is a linear combination of the other vectors in S, and one set T which spans V but is also linearly independent. (I.e. T should be a basis for V.) Follow the proof of (ii) through on your two examples and see how it works on your specific S, and what goes wrong on your T.

6.22. Look at the proof of "Check one get one free for bases".

a) What proof technique is used in the proof of both parts? (E.g. direct proof, proving the contrapositive, proof by contradiction, proof by induction, ...)

b) Explain the main ideas of the proofs of the two parts in your own words.

c) Explain how this result is used in Examples 6.14 to save on calculation when showing that the given set is a basis.

6.23. Look at the proof of Theorem 6.15, "extend to basis".

a) For both parts separately, draw a concept map of which results and definitions are needed in the proof (with names and numbers).

b) Draw in links between these results and definitions: e.g. one result used here might also need in its own proof another result used in this proof.

c) In each place a previous result or definition is needed, use the self-explanation technique to write down a self-explanation of how/why this result is being used or needed in this step.

6.24. Go through the proofs from Chapter 6. Wherever it says "similarly" or something of that kind, write out/fill in the steps that are being summarised.

Stretch yourself

6.25. Prove the following, called the **Steinitz exchange lemma**: Let V be a vector space, and let $S = \{v_1, v_2, \ldots, v_m\}$ be a linearly independent set in V, and $T = \{w_1, w_2, \ldots, w_k\}$ be a spanning set of V. Then it is possible to replace m elements from T by the elements of S, such that the resulting set still spans V. (i.e., possibly after renumbering, the set $\{v_1, \ldots, v_m, w_{m+1}, \ldots, w_k\}$ spans V.)

Hints:

a) Use the Plus/Minus Theorem to show that you can remove some vector from T, wlog w_1, such that $\{v_1, w_2, \ldots, w_k\}$ still spans V.

b) If you now do it again, explain why you will never have to remove one of the v_i that you have already put into the set, but can always choose some w_i to remove.

c) Explain why you don't run out of ws before you run out of vs.

Note that this gives you slightly more control over how you want to extend some linearly independent set to a basis than is clear from your proof of "Extend to basis" in the notes:

if S is your linearly independent set, then you can choose your favourite basis as T (e.g. the standard basis), and only add vectors from that special set (rather than just any old vectors). This is in fact what we have described in the algorithm on how to extend to a basis in the notes.

Bases and Dimension of subspaces

Introductory exercises

6.26. Let $W = \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$ and $V = \mathbb{R}^3$.

Choose a basis of W and a basis V which contains that basis of W.

Give basis containing subspace basis in Numbas.

Essential practice

6.27. Recall that $\mathcal{M}_{n,n} = \mathrm{Sym}_n \oplus \mathrm{ASym}_n$: every matrix can be uniquely written as the sum of a symmetric matrix and an anti-symmtric matrix.

a) Write down a basis for Sym_2 and a basis for ASym_2. Prove that the union of these bases forms a basis for $\mathcal{M}_{2\times 2}$.

b) Generalise your result to any n, rather than $n = 2$.

6.28. Prove Theorem 6.20(ii), i.e. prove:

If W is a subspace of a finite-dimensional vector space V, then $\dim W \leq \dim V$.

You may assume that W is finite-dimensional. Note that a basis of W is a linearly independent set in V.

6.29. Let $V = \mathbb{R}^4$, and

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \; x_1 + x_2 + x_3 = 0 \right\}, \quad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \; x_1 + 2x_2 + x_3 = 0 \right\}.$$

In the lecture notes, we have worked out that $U \cap W$ has basis $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Write down bases of U and of W that contain this basis of $U \cap W$, and write down a basis of $U + W$.

Find bases of sum given with equations in Numbas.

6.30. Let $V = \mathbb{R}^4$. In each case, find a basis for $U \cap W$, bases for U and for W which contain your basis of $U \cap W$, and a basis for $U + W$. Also give the dimension of each space.

a) $$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_1 + x_2 + x_3 + x_4 = 0 \text{ and } x_1 + 2x_2 + x_3 + 2x_4 = 0 \right\}$$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \text{ and } 2x_1 + 3x_2 + 2x_3 + 3x_4 = 0 \right\}$$

Find bases of intersection and sum, given equations, part a) in Numbas.

b) $$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_1 + 2x_2 + x_3 - 3x_4 = 0 \right\}$$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_1 - x_2 + 4x_3 - 3x_4 = 0 \right\}$$

Find bases of intersection and sum, given equations, part b) in Numbas.

6.31. In each case, find a basis for $U \cap W$, and find the dimension of $U + W \le V$. You may assume that the vectors giving U are linearly independent (and those giving W also).

a) $V = \mathbb{R}^4$,

$$U = \left\langle u_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\rangle, W = \left\langle w_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -3 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 3 \end{pmatrix} \right\rangle$$

Find bases of intersection and sum, given span, part a) in Numbas.

b) $V = \mathbb{R}^4$,

$$U = \left\langle u_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -4 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\rangle, W = \left\langle w_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 4 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} -1 \\ 3 \\ 5 \\ -4 \end{pmatrix} \right\rangle$$

Find bases of intersection and sum, given span, part b) in Numbas.

Chapter 7. Complex numbers

Introductory exercises

7.1. Given the complex numbers $z_1 = 3 - 4i$, $z_2 = 2 + 3i$, calculate

a) $z_1 + z_2$, b) $z_1 \cdot z_2$, c) $z_1 \cdot \overline{z_1}$, d) $\dfrac{z_2}{|z_1|}$.

Complex number calculations in Numbas. Randomised complex number calculations.

Essential practice

7.2. For $z = -3 + 2i$, draw (and label) the following on an Argand diagram:

z, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\overline{z}$.

7.3. a) Prove that a complex number z is real if and only if $z = \overline{z}$.

b) Prove that $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z_2}}{|z_2|^2}$. Conclude that to divide by a complex number, we only have to multiply complex numbers and divide real and imaginary parts by a real number.

7.4. See links on blackboard (in the Chapter 7 folder) for more practice opportunities on complex numbers.

7.5. Using $w\overline{w} = |w|^2$, show that the equation

a) $z\overline{z} - 4 = 0$ describes a circle in the Argand plane with centre 0 and radius 2;

b) $z\overline{z} - 3z - 3\overline{z} + 9 = 4$ describes a circle in the Argand plane with centre 3 and radius 2.

7.6. Let $u \in \mathbb{C}^n$ and let A be a $m \times k$ and B a $k \times n$ complex matrix. Prove that

a) $\overline{\overline{u}} = u$,

b) $\overline{(AB)} = (\overline{A})(\overline{B})$.

c) If you want more practice, prove the other points from "Properties of complex conjugation", Proposition 7.6.

7.7. a) Prove that the set of complex numbers $\mathbb{C}$ is a real vector space. (You should check all axioms.)

b) Give the definition of a basis of a vector space, and prove that $z_1 = 1$ and $z_2 = i$ form a basis of $\mathbb{C}$ as a real vector space.

c) Are $z_3 = 1 + i$ and $z_4 = 1 - i$ linearly independent in the real vector space $\mathbb{C}$? Do they form a basis of $\mathbb{C}$ as a real vector space?

7.8. Find the roots of the following polynomials:

a) $x^2 + 4x + 5$

b) $x^2 + 2x + 10$

c) $x^3 + 7x^2 + 33x - 41$

Factorise polynomials in Numbas.

Stretch yourself

7.9. a) Express $I = \dfrac{z^5 - 1}{z - 1}$ as a polynomial in z.

b) Given that $\omega = e^{i\frac{2\pi}{5}}$ is the complex number with modulus $|\omega| = 1$ and argument $\arg\omega = \frac{2\pi}{5}$, find the four complex roots of I. (This ω is called a **5th root of unity**, because $\omega^5 = 1$.)

c) Use this to write I as the product of two real quadratics.

d) Consider the term with z^2 in the equation you have obtained for I through the previous parts to prove that $4\cos\dfrac{\pi}{5}\sin\dfrac{\pi}{10} = 1$.

7.10. Consider the triangle made up out of the points 0, z_1 and z_2 in the Argand plane.

a) Explain why the equation $z = t\dfrac{z_1 + z_2}{2}$, as we vary $t \in \mathbb{R}$ from 0 to 1, describes the line connecting the origin with the midpoint of the side connecting z_1 and z_2, i.e. a median of the triangle.

b) Explain why $z = t\dfrac{z_1}{2} + (1 - t)z_2$, as we vary $t \in [0, 1]$, describes the median through vertex z_2. Give the equation of the same line in the form $z = a + tb$ for complex numbers a and b.

c) Write down the equation for the third median of the triangle.

d) Prove using the above that the medians of the triangle meet in one point.

Chapter 8. Linear Maps

A-level recap

8.1. If $f(x) = 5x + 1$,

a) what is $f(3a)$? b) What is $f(2u + 7v)$?

Values of f in Numbas. Randomised values of f.

8.2. If $g(x, y) = 2x - 3y$,

a) what is $g(3a, 2b)$? b) What is $g(ku_1 + lv_1, ku_2 + lv_2)$?

Values of g in Numbas. Randomised values of g.

Definition and basic properties

Introductory exercises

8.3. Let V and W be vector spaces. Show that:

a) the zero map $0 \colon V \longrightarrow W$ which sends any vector $v \in V$ to $0 \in W$ is a linear map.

b) the identity map $\mathrm{id} \colon V \longrightarrow V$ which sends any $v \in V$ to itself is a linear map.

Essential practice

8.4. a) Show that $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 4x_1 - x_2 \\ x_1 + 2x_2 \end{smallmatrix}\right)$ is a linear map.

b) Show that $T \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} x_1 + x_2 - x_3 \\ x_1 - 2x_3 \end{smallmatrix}\right)$ is a linear map.

8.5. Determine whether the following maps are linear or not.

a) $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 4 + x_1 \\ x_1 - 9x_2 \end{smallmatrix}\right)$.

b) $T \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 3x_2 \\ 5x_1 \end{smallmatrix}\right)$.

c) $T \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} x_1 + x_2 \\ 0 \end{smallmatrix}\right)$.

d) $T \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T\left(\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 3x_1 + 2x_3 \\ x_1 \cdot x_3 \end{smallmatrix}\right)$.

Determine linearity in Numbas.

8.6. Show that if $T \colon V \longrightarrow W$ is a linear map, then

$$T(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = \lambda_1 T(v_1) + \lambda_2 T(v_2) + \cdots + \lambda_n T(v_n).$$

(i.e. show that if we know T preserves linear combinations of two vectors, then it also preserves linear combinations of many vectors.)

8.7. Show that if $T \colon V \longrightarrow W$ is a linear map, then

(i) $T(0) = 0$ and (ii) $T(u - v) = T(u) - T(v)$.

(I.e. prove Proposition 8.7.)

8.8. Determine whether the following are linear maps.

a) $T\colon M_{22} \longrightarrow M_{23}$, where B is a fixed 2×3 matrix and $T(A) = AB$.

b) $T\colon M_{nn} \longrightarrow \mathbb{R}$, where $T(A) = \operatorname{tr}(A)$.

c) $F\colon M_{mn} \longrightarrow M_{nm}$, where $F(A) = A^T$.

d) $T\colon P_2 \longrightarrow P_2$ where $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$.

e) $T\colon P_2 \longrightarrow P_2$ where $T(a_0 + a_1x + a_2x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$.

8.9. Prove the result on "Composition with identity",

i.e. show that for any linear map $T\colon U \longrightarrow V$, we have $T \circ \mathrm{id}_U = T\colon U \longrightarrow V$ and

$\mathrm{id}_V \circ T = T\colon U \longrightarrow V$.

Hint: What is $T \circ \mathrm{id}_U(u)$ for any $u \in U$? Is it the same as $T(u)$?

8.10. Let $v_1 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$. You may take for granted that this is a basis for $\mathbb{R}^2$. Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation such that

$$T(v_1) = \begin{pmatrix}1\\2\\0\end{pmatrix} \qquad \text{and} \qquad T(v_2) = \begin{pmatrix}0\\3\\5\end{pmatrix}.$$

Find a formula for $T\left(\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)\right)$ and use that formula to find $T\left(\left(\begin{smallmatrix}2\\-3\end{smallmatrix}\right)\right)$.

Find formula for linear map in Numbas. Randomised formula for linear map.

8.11. If $T, S\colon U \longrightarrow V$ are two linear maps, show that the map $T + S\colon U \longrightarrow V$ defined by $(T+S)(u) = T(u) + S(u)$ is linear.

Stretch yourself

8.12. Build on Question 8.11 to show that the set of linear maps from V to W forms a vector space, which we call $\mathcal{L}(V, W)$.

8.13. We have seen that differentiation is a linear map.

a) Show that "taking a definite integral" $f(x) \longmapsto \int_0^1 f(t)dt$ is a linear map $C[0,1] \longrightarrow \mathbb{R}$ from continuous functions on $[0,1]$ to $\mathbb{R}$.

b) Show that "taking an anti-derivative" $f(x) \longmapsto F(x) = \int_0^x f(t)dt$ is a linear map $C[0,1] \longrightarrow C[0,1]$.

(This is saying two things: integration preserves linear combination, and if you take this integral of a continuous function, you get another continuous function.)

c) Think about the differences between these two linear maps: consider their codomains.

d) Have a look in your Probability (or Statistics) notes to see that taking the expectation of a random variable is a linear map, but that variance is not linear. What are the domain and codomain of "taking the expectation of a random variable"? Can you show that they are vector spaces?

Kernels, Images, Surjective/Injective Functions, Isomorphisms

Introductory exercises

8.14. Determine the kernel and image of

a) the zero map $0\colon V \longrightarrow W$ and b) the identity $\mathrm{id}\colon V \longrightarrow V$.

Kernel and image of 0 and id in Numbas.

8.15. Let P_n be the space of complex polynomials of degree $\leq n$, and let $S\colon P_n \longrightarrow \mathbb{C}^{n+1}$ be

$$S(a_0 + a_1 x + \cdots + a_n x^n) = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

a) Show that S is a linear map.

b) Show that S is injective.

c) Show that S is surjective.

Essential practice

8.16. Let $T\colon V \longrightarrow W$ be a linear map.

a) Define the *kernel* and *image* of T.

b) Define what it means for T to be *injective*, *surjective* or an *isomorphism*.

8.17. Let $T_A\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the matrix transformation $T_A(v) = Av$ given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

a) Find bases for the image and the kernel of T.

b) Find the rank and nullity of T.

Kernel, image, rank, nullity in Numbas. Kernel, image, rank, nullity randomised.

8.18. In each part, find a basis for the kernel and for the image, and determine whether the map is injective, surjective, an isomorphism. For each isomorphism, find its inverse.

a) $T_A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $A = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$.

Kernel, image, inverse part a) in Numbas.

b) $T_B\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $B = \begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix}$.

Kernel, image, inverse part b) in Numbas.

c) $T_C\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with $C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

Kernel, image, inverse part c) in Numbas.

d) $T_D\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$.

Kernel, image, inverse part d) in Numbas.

Kernel, image, inverse randomised.

8.19. For each matrix, determine the domain (source) and codomain (target) of the matrix transformation T_A which multiplies a vector by A, find a basis for the kernel (if the kernel is not 0) and for the image of T_A, and determine whether T_A is injective, surjective, an isomorphism.

a) $A = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{pmatrix}$ Domain, Kernel, etc of part a).

b) $A = \begin{pmatrix} 1 & 3 & 5 & 7 \\ -1 & 3 & 0 & 0 \\ 1 & 15 & 15 & 21 \end{pmatrix}$ Domain, Kernel, etc of part b).

c) $A = \begin{pmatrix} 2 & 5 \\ -1 & 3 \\ 2 & 4 \end{pmatrix}$ Domain, Kernel, etc of part c).

8.20. Let $T\colon U \longrightarrow V$ be a linear map.

Prove that $\operatorname{Im} T$ is a subspace of the codomain V.

8.21. Let $T\colon V \longrightarrow V$ be a linear map. Show that

a) $\operatorname{Ker}(T \circ T) \supseteq \operatorname{Ker}(T)$ and

b) $\operatorname{Im}(T \circ T) \subseteq \operatorname{Im}(T)$.

8.22. State and prove a condition for a linear map $T\colon V \longrightarrow W$ to be injective which involves the kernel of T.

8.23. Given linear maps $T\colon U \longrightarrow V$ and $S\colon V \longrightarrow U$, show that if $S \circ T = \mathrm{id}_U$, then

(i) T is injective, and

(ii) S is surjective.

8.24. Use the self-explanation technique to work through the proof of: a linear map $T\colon V \longrightarrow V$ is injective if and only if it is surjective.

Also: give an example of a linear map $T\colon V \longrightarrow W$ which is injective but not surjective, and one which is surjective and not injective. (You should choose V and W yourself.)

8.25. Suppose V, W are finite-dimensional with $\dim W < \dim V$.

a) Prove that there cannot be any injective linear map $T\colon V \longrightarrow W$.

b) Prove that there cannot be any surjective linear map $S\colon W \longrightarrow V$.

8.26. Look at the proof of "isomorphisms preserve bases".

a) What previous results of the course are used in this proof, and where?

b) Pick an example of an isomorphism $T\colon V \longrightarrow W$ (pick V and W as well), and pick a basis of V, and work out which basis of W the basis you picked is mapped to by T.

c) Pick a homomorphism $S_1\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which is surjective but not injective. What can you say about $S_1(e_1)$, $S_1(e_2)$, $S_1(e_3)$, the images of the standard basis? I.e. are they a basis of $\mathbb{R}^2$? If not, which conditions fail?

d) Pick a homomorphism $S_2\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which is injective but not surjective. What can you say about $S_2(e_1)$, $S_2(e_2)$, the images of the standard basis? I.e. are they a basis of $\mathbb{R}^3$? If not, which conditions fail?

8.27. For each pair of vector spaces V and W below, say if there exists some isomorphism $V \longrightarrow W$. Justify your answer carefully each time. If the answer is "yes", give an isomorphism $V \longrightarrow W$ and its inverse (you do not have to prove that it is an isomorphism).

a) $V = \mathbb{R}^6$, $W = \mathbb{R}^8$, both real vector spaces.

b) $V = P_n$, $W = \mathbb{R}^{n+1}$, both real vector spaces.

c) $V = P$ (all real polynomials), $W = \mathbb{R}^4$, both real vector spaces.

d) $V = \mathbb{C}^2$, $W = \mathbb{R}^4$, both as real vector spaces.

e) $V = \mathbb{C}^{\frac{1}{2}(n^2+n)}$, $W = \{\text{complex symmetric matrices of size } n\times n\}$, both complex vector spaces.

f) $V = \{\text{real anti-symmetric matrices of size } n \times n\} = \{A \mid A^T = -A\}$, $W = \mathbb{R}^{n^2-n}$, both real vector spaces.

8.28. Let $T = \mathrm{id}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the identity map on $\mathbb{R}^2$, and let $S = -\mathrm{id}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Recall from Question 8.11 that then $T + S$ is also a linear map. In b) and c), justify your answer carefully (i.e. give a proof or counterexample).

a) Find the subspaces $\mathrm{Im}(T + S)$, $\mathrm{Im}(T) + \mathrm{Im}(S)$ and $\mathrm{Ker}(T + S)$, $\mathrm{Ker}(T) \cap \mathrm{Ker}(S)$ of $\mathbb{R}^2$.

b) Is it true that for any $u, v \in \mathbb{R}^2$ there is some $w \in \mathbb{R}^2$ such that $T(u) + S(v) = (T + S)(w)$?

c) Is it true that for any $v \in \mathbb{R}^2$ with $(T + S)(v) = 0$ we must have $T(v) = 0 = S(v)$? [I.e. is it true that $\mathrm{Ker}(T + S) \subseteq \mathrm{Ker}(T) \cap \mathrm{Ker}(S)$?]

Stretch yourself

8.29. Let $T, S\colon V \longrightarrow W$ be two linear maps. Show that

a) $\operatorname{Im}(T+S) \leq \operatorname{Im}(T) + \operatorname{Im}(S)$;

b) $\operatorname{Ker}(T) \cap \operatorname{Ker}(S) \leq \operatorname{Ker}(T+S)$.

c) Use Question 8.28 to show that there are examples of T and S where these are proper subspaces (i.e. not equal).

d) Give an example of T, S, V, W where $\operatorname{Im}(T+S) = \operatorname{Im}(T) + \operatorname{Im}(S)$ and an example where $\operatorname{Ker}(T) \cap \operatorname{Ker}(S) = \operatorname{Ker}(T+S)$.

8.30. Let $T\colon V \longrightarrow V$ be a linear map, where V is a finite-dimensional vector space. Show that if $T^2 = T$, then $V = \operatorname{Ker}(T) \oplus \operatorname{Im}(T)$.

Matrix represenation of linear maps, base change

Introductory exercises

8.31. In each part, give the matrix with respect to the standard bases for all spaces.

a) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $T\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = \begin{pmatrix} 2y \\ 3x \end{pmatrix}$.

b) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $T\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = \begin{pmatrix} 5x-y \\ 0 \end{pmatrix}$.

c) $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x-z \end{pmatrix}$.

d) $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with $T\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) = \begin{pmatrix} y \\ x \\ x-y \end{pmatrix}$.

Matrices wrt standard bases in Numbas. Randomised matrices wrt standard bases.

Essential practice

8.32. Write out the definition of a matrix representing a linear map with respect to given bases.

8.33. Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be linear with $T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$, $T\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)$ and $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right)$.

a) Determine the matrix for T with respect to the bases $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\mathbb{R}^3$ and the standard basis for $\mathbb{R}^2$.

b) Determine the matrix for T with respect to the bases v_1, v_2, v_3 for $\mathbb{R}^3$ and $w_1 = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right), w_2 = \left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)$ for $\mathbb{R}^2$.

c) Determine the matrix for T with respect to the standard bases for both $\mathbb{R}^3$ and $\mathbb{R}^2$.

Matrices for linear map in Numbas.

Slightly different version of finding a matrix for linear map wrt different bases, randomised.

8.34. Let $T\colon P_2 \longrightarrow P_1$ be the linear map with $T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) - (2a_1 + 3a_2)x$.

a) Find the matrix for T with respect to the standard bases E: 1, x, x^2 for P_2 and E': 1, x for P_1.

b) Verify that the matrix ${}_{E'}[T]_E$ you have found satisfies the formula $[T(p)]_{E'} = {}_{E'}[T]_E[p]_E$ (from Lemma 8.42 "linear maps as matrix transformations") for any polynomial $p = a_0 + a_1x + a_2x^2$.

First matrix for polynomial map in Numbas.

8.35. Let $T\colon P_2 \longrightarrow P_2$ be the linear map with $T(a_0+a_1x+a_2x^2) = a_0+a_1(x-1)+a_2(x-1)^2$.

a) Find the matrix for T with respect to the standard basis $E = \{1, x, x^2\}$ for P_2.

b) Verify that the matrix $[T]_E$ you have found satisfies the formula $[T(p)]_E = [T]_E[p]_E$ (from Lemma 8.42 "linear maps as matrix transformations") for any polynomial $p = a_0 + a_1x + a_2x^2$.

Second matrix for polynomial map in Numbas.

8.36. a) Determine the coordinate vectors for the polynomials $p_1 = -1 + x - 2x^2$, $p_2 = 3 + 3x + 6x^2$, $p_3 = 9$ with respect to the basis E: 1, x, x^2 for P_2.

b) The linear map $S\colon P_3 \longrightarrow P_2$ is defined by

$$S(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0p_1 + a_1p_2 + (a_2 + a_3)p_3.$$

Determine the matrix for S with respect to the standard bases $1, x, x^2, x^3$ for P_3 and $1, x, x^2$ for P_2.

c) Determine the matrix for S with respect to the standard basis for P_3, but the basis p_1, p_2, p_3 for P_2.

Matrices of polynomial maps with different bases in Numbas.

8.37. Let V be a finite dimensional vector space with bases B and C, and let $T\colon V \longrightarrow V$ be a linear map. Let $A = [T]_B$ be the matrix for T with respect to basis B, and $D = [T]_C$ be the matrix for T with respect to basis C. Let $P = {}_C[\mathrm{id}_V]_B$. Prove that $PA = DP$ and conclude that $D = PAP^{-1}$.

[Hint: use the result called "matrix for composite is product of matrices".]

8.38. a) Use the self-explanation technique to go through the proof of "similar matrices have the same determinant".

b) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, by looking at elements.

c) Prove that similar matrices have the same trace.

8.39. Consider the bases B: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\mathbb{R}^3$, C: $w_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, w_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ for $\mathbb{R}^2$, and the two standard bases E_3 for $\mathbb{R}^3$ and E_2 for $\mathbb{R}^2$.

a) Determine the base change matrices $P_{B \longrightarrow E_3}$ and $P_{E_3 \longrightarrow B}$.

b) Determine the base change matrices $P_{C \longrightarrow E_2}$ and $P_{E_2 \longrightarrow C}$.

c) Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be linear with $T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $T\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. In Question 8.33, you determined the matrices ${}_{E_2}[T]_B$, ${}_C[T]_B$ and ${}_{E_2}[T]_{E_3}$. Verify that $P^{-1}_{C \longrightarrow E_2} {}_{E_2}[T]_{E_3} P_{B \longrightarrow E_3} = {}_C[T]_B$.

Base change in Numbas. Randomised base change without part c).

Stretch yourself

8.40. Let $U \leq V$ and $X \leq W$ be subspaces of two finite-dimensional vector spaces V and W. Let $T\colon V \longrightarrow W$ be a linear map such that $T(U) \subseteq X$, i.e. for any $u \in U$, $T(u) \in X$.

a) Let $v_1, \ldots, v_k$ be a basis of U, extended to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ of V. Let $w_1 \ldots, w_l$ be a basis of X, extended to a basis $w_1, \ldots, w_l, w_{l+1}, \ldots, w_m$ of W. Show that the matrix of T with respect to the given bases for V and W is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A has size $k \times l$ and C has size $(n-k) \times (m-l)$.

b) Recall from Question 8.12 that we have a vector space $\mathcal{L}(V, W)$ of linear maps from V to W. Show that the set of linear maps $\{T\colon V \longrightarrow W \mid T(U) \subseteq X\}$ is a subspace of $\mathcal{L}(V, W)$.

c) Use a) to find the dimension of the subspace in b).

Chapter 9. Eigenvectors and Eigenvalues

Introductory exercises

9.1. Find the characteristic polynomial, eigenvalues and eigenvectors (bases for eigenspaces) for the following matrices:

a) $\begin{pmatrix} 2 & 0 \\ 0 & -9 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a) Intro evalues and evectors in Numbas. b) Evalues and evectors of zero matrix.

c) Evalues and evectors of identity matrix. d) Randomised intro evalues and evectors.

Essential practice

9.2. Define the terms *eigenvector* and *eigenvalue* of a linear map $T\colon V \longrightarrow V$.

9.3. Find the characteristic polynomial, eigenvalues and eigenvectors (bases for eigenspaces) for the following matrices:

a) $\begin{pmatrix} 2 & -1 \\ 10 & -9 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 3 \\ 12 & 0 \end{pmatrix}$ c) $\begin{pmatrix} -2 & 5 \\ 1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 4 & -2 \\ 12 & -6 \end{pmatrix}$

2×2 evalues and evectors a). 2×2 evalues and evectors b). 2×2 evalues and evectors c). 2×2 evalues and evectors d). 2×2 evalues and evectors randomised.

9.4. Find the characteristic polynomial, eigenvalues and eigenvectors (bases for eigenspaces) for the following matrices:

a) $\begin{pmatrix} 5 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 6 & 12 \\ 0 & 3 & 10 \\ 0 & 0 & 3 \end{pmatrix}$ c) $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$ d) $\begin{pmatrix} 5 & 6 & 2 \\ -1 & 0 & -1 \\ 2 & 6 & 5 \end{pmatrix}$

e) $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

3×3 evectors a). 3×3 evectors b). 3×3 evectors c). 3×3 evectors d).

4×4 evectors e). 4×4 evectors f).

9.5. Using the matrices from Question 9.3b) and 9.3d), determine which of the matrices are invertible. (Use your knowledge of the eigenvalues for this.) For each matrix which is invertible, find the eigenvalues and eigenvectors of the inverse.

Determine invertibility b), determine invertibility d), determine invertibility randomised.

9.6. Draw a concept map of all results that go into the proof of: $T\colon V \longrightarrow V$ is an isomorphism if and only if 0 is not an eigenvalue of T. How far back can you trace the dependency of these results, into earlier chapters?

9.7. Prove that λ is an eigenvalue of the matrix A if and only if $\det(\lambda I - A) = 0$.

9.8. Explain both conceptually and algebraically why similar matrices have the same characteristic polynomial.

9.9. Let $A \in \mathcal{M}_{n,n}$, with characteristic polynomial $\chi_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. Prove that $a_0 = (-1)^n \det(A)$.

9.10. For each matrix below,

give the algebraic and geometric multiplicities of all eigenvalues. Then determine whether the matrix is diagonalisable. *(b) and d) are from Question 9.3, so you already know the eigenvalues and eigenvectors. a) and c) are upper triangular/diagonal, so eigenvalues and eigenvectors will be easy to calculate. Numbas gives them to you, so if you want to do it yourself, do that before you open the Numbas question.)*

a) $\begin{pmatrix} 3 & 15 \\ 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 3 \\ 12 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ d) $\begin{pmatrix} 4 & -2 \\ 12 & -6 \end{pmatrix}$

2×2 algebraic and geometric mult a). 2×2 algebraic and geometric mult b).
2×2 algebraic and geometric mult c). 2×2 algebraic and geometric mult d).

9.11. For each matrix from Question 9.4,

give the algebraic and geometric multiplicities of all eigenvalues. Then determine whether the matrix is diagonalisable.

3×3 algebraic and geometric mult a). 3×3 algebraic and geometric mult b).
3×3 algebraic and geometric mult c). 3×3 algebraic and geometric mult d).
4×4 algebraic and geometric mult e). 4×4 algebraic and geometric mult f).

9.12. a) Write down a sufficient condition for T to be diagonalisable which involves only the eigenvalues of T.

b) Write down a necessary and sufficient condition for T to be diagonalisable which involves the eigenvectors of T.

c) Write down a necessary and sufficient condition for T to be diagonalisable which involves the algebraic and geometric multiplicities of T.

9.13. Give examples of a 3×3 matrix which

a) is diagonalisable but not diagonal.

b) is not diagonalisable but has at least 2 real eigenvalues.

c) does not have enough real eigenvalues.

9.14. Consider the matrices

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

a) Find the characteristic polynomials for A and B, and their eigenvalues.

Diagonal vs upper triangular eigenvectors in Numbas.

b) If P is an invertible matrix, what is $P^{-1}AP$?

c) Suppose there is a basis for $\mathbb{R}^2$ consisting of eigenvectors of B, and let Q be the matrix with columns being those eigenvectors. What form would $Q^{-1}BQ$ have to take?

d) Combine the previous parts to deduce that there cannot be a basis of eigenvectors for B.

9.15. The matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

has eigenvalues 2 and 1.

a) Find the eigenvectors of A.

b) What are the evalues and evectors of A^7? And of $A^7 + A^2$?

c) If P is the matrix with these eigenvectors as columns, then $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D$.

What is $P^{-1}A^7P$? And $P^{-1}(A^7 + A^2)P$?

Evectors of powers in Numbas.

9.16. The matrix $A = \begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$.

a) Find eigenvectors v_1 for λ_1 and v_2 for λ_2.

b) Let P be the matrix with columns v_1 and v_2, in that order. Determine $D = P^{-1}AP$.

[Hint: you can do this by doing all the calculations, or you can do it by thinking, without much calculation.]

c) Let $C = A^{13} + A^{23}$. Is C diagonalisable? If yes, give its diagonal form and the corresponding base change matrix. (You are allowed to write integers as sums of powers, you don't have to work out the value.)

Diagonalisability of powers in Numbas.

9.17. a) Prove that A and A^T have the same characteristic polynomial.

b) Deduce that A and A^T have the same eigenvalues, with same algebraic multiplicities.

c) Find the eigenvalues and eigenvectors of A and A^T,

$$\text{with} \qquad A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

9.18. Let

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

a) Consider R as a real matrix, so as a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. What does this map do geometrically? Find the eigenvalues.

b) Now consider R as a complex matrix, so as a linear map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$. Find the eigenvalues.

c) For all eigenvalues you found in the previous two parts, find eigenvectors.

d) Can you find a basis for $\mathbb{R}^2$ which consists of (real) eigenvectors of the matrix R? If yes, give it.

e) Can you find a basis for $\mathbb{C}^2$ which consists of (complex) eigenvectors of the matrix R? If yes, give it.

9.19. Construct a non-zero 3×3 real matrix which has 0 as its only eigenvalue.

Check matrix with only 0 evalue in Numbas.

Stretch yourself

9.20. Let A be an invertible $n \times n$ matrix. Let $\chi_A(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of A. Show that the characteristic polynomial of A^{-1} is

$$\chi_{A^{-1}}(t) = (-1)^n \det(A^{-1})(1 + a_{n-1}t + \cdots + a_1 t^{n-1} + a_0 t^n).$$

Use Question 9.9 to show that this characteristic polynomial of A^{-1} is also monic (meaning the coefficient of t^n is 1).

9.21. Find a real 3×3 matrix with eigenvalues $1,\ i,\ -i$.

[Hint: think geometrically.]

9.22. Let A be a square matrix such that $A^m = 0$ for some positive integer m. Prove that every eigenvalue of A is 0.

Chapter 10. Inner Products

Standard (Euclidian) inner product

Introductory exercises

10.1. Work out $\langle v, w\rangle = v^T w$ and $\|v\|$ in each example, using the Euclidean inner product.

a) For $v = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$, $w = \begin{pmatrix} -4 \\ 7 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$.

b) For $v = \begin{pmatrix} -1 \\ 5 \\ 4 \\ 4 \end{pmatrix}$, $w = \begin{pmatrix} -4 \\ -2 \\ 4 \\ 3 \end{pmatrix}$ in $\mathbb{R}^4$.

Intro inner product and norm in Numbas, randomised.

10.2. For each vector v, give the normalised vector $u = \frac{1}{\|v\|}v$.

a) $v = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ b) $v = \begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix}$ c) $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ d) $v = \begin{pmatrix} -2 \\ 1 \\ -4 \\ 2 \end{pmatrix}$

Normalise vectors in Numbas. Random normalisation $\mathbb{R}^3$. Random normalisation $\mathbb{R}^4$.

10.3. Determine if v and w are orthogonal.

a) $v = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$, $w = \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$ b) $v = \begin{pmatrix} -4 \\ 8 \\ -7 \end{pmatrix}$, $w = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$

c) $v = \begin{pmatrix} -1 \\ 5 \\ 4 \\ 4 \end{pmatrix}$, $w = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ d) $v = \begin{pmatrix} -1 \\ 5 \\ 4 \\ 4 \end{pmatrix}$, $w = \begin{pmatrix} -9 \\ -1 \\ 2 \\ -3 \end{pmatrix}$

Determine orthogonality in Numbas. Randomised orthogonality in Numbas.

Essential practice

10.4. Prove that the standard norm on $\mathbb{R}^n$ satisfies:

(i) $\|v\| \geq 0$.

(ii) $\|v\| = 0 \iff v = 0$.

(iii) $\|\lambda v\| = |\lambda|\|v\|$.

I.e. prove Corollary 10.6, Properties of norm. Use the inner product properties. Remember $\|v\|^2 = \langle v, v\rangle$.

10.5. Given $v \in \mathbb{R}^n$, prove that the set of vectors which is orthogonal to v forms a subspace of $\mathbb{R}^n$. I.e. prove that

$$\{w \in \mathbb{R}^n \mid \langle v, w\rangle = 0\} \leq \mathbb{R}^n.$$

10.6. Find the vectors which are orthogonal to $\begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \\ 4 \end{pmatrix} \in \mathbb{R}^5$.

Hint: use an equation like we did in lectures.

Find orthogonal vectors in Numbas.

10.7. This question is meant to help you with the proof of the Cauchy-Schwarz inequality.

Let $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

a) Calculate $\langle u, v\rangle$ and $\|u\|^2$, $\|v\|^2$ and $\|u\|^2\|v\|^2$.

b) Calculate $\langle u+tv, u+tv\rangle$, i.e. write out the polynomial in t that this gives you. Link back the coefficients you get to the expressions you got for $\langle u, v\rangle$ and $\|u\|$, $\|v\|$.

c) Work out the discriminant of the polynomial you have just found.

d) Which property of the inner product tells us that $\langle u + tv, u + tv\rangle \geq 0$?

e) What does this tell us about the discriminant?

f) When is the discriminant 0?

g) Give examples for v such that

- the discriminant is 0 and $\langle u, v\rangle > 0$,
- the discriminant is 0 and $\langle u, v\rangle < 0$,
- the discriminant is non-zero.

h) For your three examples, compare $\langle u, v\rangle$ and $\|u\|\|v\|$ (or their squares).

i) If you would like more understanding, repeat the question in $\mathbb{R}^4$, with $u = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Or you can pick a different u.

10.8. In the proof of the triangle inequality,

fill in the detailed steps for why

$$\langle v + w, v + w\rangle = \|v\|^2 + 2\langle v, w\rangle + \|w\|^2.$$

You should be very careful and have at least two steps in between these two expressions.

Stretch yourself

10.9. Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map.

We define the **adjoint** T^* of T to be the linear map which satisfies $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$ for any $v, w \in \mathbb{R}^n$, using the standard inner product. Suppose A is the matrix for T with respect to the standard basis.

a) What matrix represents T^* with respect to the standard basis?

b) What property should A have such that $T = T^*$, i.e. T is **self-adjoint**?

10.10. Let $T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a matrix transformation.

Prove that if $A^TA = I$, then T_A preserves lengths: $\|T_A(v)\| = \|v\|$.

General inner product

Introductory exercises

10.11. Work out $\langle v, w\rangle = v^T D w$ and $\|v\|$ in each example, using the weighted inner product with the given D.

a) For $v = \left(\begin{smallmatrix} -4 \\ 8 \\ -7 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} -4 \\ 7 \\ 2 \end{smallmatrix}\right)$ in $\mathbb{R}^3$, with $D = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{smallmatrix}\right)$.

b) For $v = \left(\begin{smallmatrix} -1 \\ 5 \\ 4 \\ 4 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} -4 \\ -2 \\ 4 \\ 3 \end{smallmatrix}\right)$ in $\mathbb{R}^4$, with $D = \left(\begin{smallmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{smallmatrix}\right)$.

Weighted inner product in Numbas. Randomised weighted inner product in $\mathbb{R}^3$. Randomised weighted inner product in $\mathbb{R}^4$.

Essential practice

10.12. For each vector v, give the normalised vector $u = \frac{1}{\|v\|} v$ with respect to the weighted inner product with $D = \left(\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{smallmatrix}\right)$.

a) $v = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ b) $v = \left(\begin{smallmatrix} 4 \\ -2 \\ -5 \end{smallmatrix}\right)$

Normalise vectors with weighted norm in Numbas. Randomised vectors to normalise with weighted norm.

10.13. Let V be the vector space of $n \times n$ matrices. Prove that

$$\langle A, B\rangle = \operatorname{tr}(A^T B)$$

is an inner product.

Remember: $\operatorname{tr}(C^T) = \operatorname{tr}(C)$. For positive definiteness, you'll have to look at actual elements.

10.14. Work out $\langle A, B\rangle$ and $\|A\|$ in each example, for the matrix inner product $\langle A, B\rangle = \operatorname{tr}(A^T B)$.

a) $A = \begin{pmatrix} 4 & 1 \\ 2 & -5 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -3 \\ -5 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} -3 & 2 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -4 & -3 \\ 1 & 2 \end{pmatrix}$

Matrix inner product in Numbas. Randomised matrix inner product.

10.15. Using the matrix inner product $\langle A, B\rangle = \operatorname{tr}(A^T B)$, determine if the following matrices are orthogonal.

a) $A = \begin{pmatrix} 4 & 1 \\ 2 & -5 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -3 \\ -5 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & -3 \\ 0 & -5 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -4 \\ 3 & 2 \end{pmatrix}$

Orthogonal with matrix inner product. Randomised orthogonal with matrix inner product.

10.16. Consider P_2 with the integral inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$.

Let $p_0 = 1$, $p_1 = x$, $p_2 = x^2$, $q_2 = \frac{1}{2}(3x^2 - 1)$.

a) Calculate $\|p_0\|$ and $\|p_1\|$ with the intergral inner product.

b) Calculate $\langle p_0, p_1\rangle$, $\langle p_0, p_2\rangle$ and $\langle p_0, q_2\rangle$.

c) Are p_0 and p_1 orthogonal? Are p_0 and p_2 orthogonal? Are p_0 and q_2 orthogonal?

Integral inner product in Numbas.

Stretch yourself

10.17. Let V be a suitable space of random variables. Prove that

$$\langle X, Y\rangle = \mathrm{E}(XY)$$

is an inner product (using appropriate properties of expectation from Probability).

10.18. Recall the integral inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$$

on $V = P_n$. Let $T\colon P_n \longrightarrow P_n$ be the linear map $T(p)(t) = (1-t^2)p''(t) - 2p'(t)$ (involving the first and second derivatives). Show that T is self-adjoint, i.e. show that $\langle T(p), q\rangle = \langle p, T(q)\rangle$ for every polynomial p and q. (C.f. Question 10.9)

Hint: you'll need to do some integration by parts.

Orthonormal bases

Introductory exercises

10.19. Using the standard inner product, determine whether the set is orthogonal but not orthonormal, an orthonormal basis, or neither.

a) The standard basis in $\mathbb{R}^n$. (Or just do e_1, e_2, e_3 in $\mathbb{R}^3$.)

b) $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^2$.

c) $v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ in $\mathbb{R}^2$.

d) $v_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$ in $\mathbb{R}^2$.

Determine orthonormal in Numbas - intro.

Essential practice

10.20. Using the standard inner product, determine whether the set is orthogonal but not orthonormal, an orthonormal basis, or neither.

a) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$.

b) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ in $\mathbb{R}^3$.

c) $v_1 = \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$, $v_2 = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -36 \\ -48 \\ 25 \end{pmatrix}$ in $\mathbb{R}^3$.

d) $v_1 = \begin{pmatrix} \frac{3}{13} \\ \frac{4}{13} \\ \frac{12}{13} \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -\frac{36}{65} \\ -\frac{48}{65} \\ \frac{5}{13} \end{pmatrix}$ in $\mathbb{R}^3$.

Determine orthonormal in Numbas.

10.21. Consider the orthonormal basis

B: $u_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$, $u_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$ in $\mathbb{R}^2$. For $v = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, determine the coordinate vector $[v]_B$ of v with respect to this orthonormal basis.

Coordinate vector for $\mathbb{R}^2$ orthonormal basis in Numbas, including randomised.

Coordinate vector for $\mathbb{R}^3$ orthonormal basis randomised.

10.22. Using the standard inner product, apply the Gram-Schmidt normalisation process to $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then also apply it to the vectors the other way around. This it to help you see that the order matters here.

Standard Gram-Schmidt in Numbas.

10.23. Using the weighted inner product with $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, apply the Gram-Schmidt normalisation process to $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(Once you've done it, you can check your workings against Example 10.27 in the notes.)

Stretch yourself

10.24. Using the integral inner product, apply the Gram-Schmidt normalisation process to the standard basis of P_2, i.e. to $p_0 = 1$, $p_1 = x$, $p_2 = x^2$.

Complex inner product

Introductory exercises

10.25. Work out the inner product and norm of some complex vectors in Numbas.

Essential practice

10.26. Prove that the complex norm on $\mathbb{C}^n$ satisfies:

(i) $\|v\| \geq 0$.

(ii) $\|v\| = 0 \iff v = 0$.

(iii) $\|\lambda v\| = |\lambda|\|v\|$.

Use the complex inner product properties.

10.27. Normalise some complex vectors to unit vectors, in Numbas.

10.28. Determine whether two complex vectors are orthogonal, in Numbas.

10.29. Let A be a real symmetric matrix.

We proved that it has only real eigenvalues when considered as a complex matrix. In the proof, we use potentially complex eigenvectors. Once we know that A has only real eigenvalues, does this mean that every such eigenvalue also has a real eigenvector, i.e. $v \in \mathbb{R}^n$ instead of $v \in \mathbb{C}^n$ as in the proof?

Stretch yourself

10.30. Let A be a complex matrix which satisfies $\overline{A}^T = A$. We call such a matrix **hermitian**. Prove that a hermitian matrix has real eigenvalues.

Hint: Use the proof "real symmetric matrix has real eigenvalues" as a guide.

10.31. Let $T_A \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a matrix transformation.

Prove that if $\overline{A}^T A = I$, then T_A preserves lengths with respect to the complex inner product: $\|T_A(v)\| = \|v\|$.