The fundamental group functor as a Kan extension

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joint work with Tomas Everaert and Tim Van der Linden

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Aim of the talk

$$
\begin{tikzcd}
& \mathsf{NExt}_\Gamma(\mathscr{C}) \arrow[dl, "\mathsf{Cod}"'] \arrow[dr, "\mathsf{Gal}_\Gamma(-{,}0)"] \arrow[d, phantom, "\overset{\kappa}{\Nearrow}"] & \\
\mathscr{C} \arrow[rr, dotted, "\pi_1(-{,}I)"'] & {} & \mathsf{Gp}(\mathscr{X})
\end{tikzcd}
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Why?

- In topological example gives some universal properties of the "usual" fundamental group, and (hopefully) also the connecting homomorphism in exact sequence induced by a fibration.
- In algebraic examples gives another approach to semi-abelian homology.

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Galois structures

Definition (Janelidze)

A Galois structure Γ consists of an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left=1.5ex, "I"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left=1.5ex, "H"] \end{tikzcd}$$

with unit η and counit ϵ, as well as classes of maps $\mathscr{E}$ in $\mathscr{C}$ and $\mathscr{F}$ in $\mathscr{X}$ satisfying certain axioms.

Examples of Galois structures

- Groups with subcategory abelian groups, regular epis.
- Semi-abelian $\mathscr{C}$ with Birkhoff subcategory, regular epis.
- Locally connected topological spaces and sets.
  H is discrete topology functor,
  $I = \pi_0$, connected components functor.
  $\mathscr{E}$ = local homeomorphisms (étale maps), $\mathscr{F}$ = all maps.
- Opposite of finite dimensional k-algebras, finite sets, each with all maps. The adjunction is defined through idempotent decomposition of k-algebras: a k-algebra is sent to its set of primitive idempotents.

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Special maps in Galois structures

- trivial coverings:
$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\mathscr{E}\ni f"'] \arrow[dr, phantom, "\lrcorner", very near start] & HIA \arrow[d, "HIf"] \\ B \arrow[r, "\eta_B"'] & HIB \end{tikzcd}$$
(*cartesian wrt. I*)

- monadic extensions: $p\colon E \longrightarrow B$ in $\mathscr{E}$ with $p^*\colon (\mathscr{E} \downarrow B) \longrightarrow (\mathscr{E} \downarrow E)$ monadic. (*good to pull back along*)

- coverings (or central extensions): $f \in \mathscr{E}$ with $p^*(f)$ trivial for some monadic p. (*locally trivial*)

- normal extensions: monadic p with trivial kernel pair projections.

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Examples

- Groups with abelian groups:
  - monadic extensions: all regular epis.
  - central extensions as usual, kernel inside centre;
- topological example:
  - monadic extensions: surjective étale maps;
  - coverings: usual topological sense;
  - normal extensions: regular coverings.
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Admissibility

From now on H is inclusion.

- admissible Galois structure: I preserves pullbacks along trivial coverings.
- When $\mathscr{E}$ is all maps,
  admissible = semi-left exact = Street fibration.
- $\Rightarrow$ Trivial coverings are pullback-stable.
- Think "trivial coverings are pullback-closure of $\mathscr{F}$ in $\mathscr{C}$".
- If monadic extensions pullback-stable, then also normal extensions and coverings pullback-stable.

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Galois groupoid (Janelidze)

Take $\mathscr{C}$ pointed and $p\colon E \longrightarrow B$ normal extension.

- Galois groupoid $\mathrm{Gal}_\Gamma(p) = I\mathrm{Eq}(p)$

$$\begin{tikzcd} \mathrm{Eq}(p) \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "\eta"'] & E \arrow[r, "p"] \arrow[d, "\eta_E"] & B \\ I\mathrm{Eq}(p) \arrow[r, shift left, "Id"] \arrow[r, shift right, "Ic"'] & IE & \end{tikzcd}$$

- Still groupoid, as I preserves defining pullbacks, because d and c trivial.
- Galois group $\mathrm{Gal}_\Gamma(p, 0) = \mathrm{Ker}\, Id \cap \mathrm{Ker}\, Ic$
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Properties of Galois group functor

Morphisms in $\mathsf{NExt}_\Gamma(\mathscr{C})$

$$\begin{tikzcd} E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$

induce

- homotopy on kernel pairs and Galois groupoids,
- same morphism $\mathrm{Gal}_\Gamma(p, 0) \longrightarrow \mathrm{Gal}_\Gamma(p', 0)$ on Galois group.
- $(f, 1_B)\colon p \longrightarrow p$ induces identity on Galois group.

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Fundamental group functor

Now assume that weakly universal normal extensions exist.

$$\begin{tikzcd} & \mathrm{NExt}_\Gamma(\mathscr{C}) \arrow[dl, "\mathrm{Cod}"'] \arrow[dr, "\mathrm{Gal}_\Gamma(-{,}0)"] \arrow[d, phantom, "\overset{\kappa}{\Uparrow}"] & \\ \mathscr{C} \arrow[rr, dotted, "\pi_1(-{,}I)"'] & & \mathrm{Gp}(\mathscr{X}) \end{tikzcd}$$

Definition (Janelidze)

Given $B \in \mathscr{C}$, pick weakly universal normal extension $u\colon U \longrightarrow B$, and let

$$\pi_1(B, I) = \mathrm{Gal}_\Gamma(u, 0).$$

Functorial in B because of induced homotopies.

Examples

- Groups and abelian groups: get $\pi_1(B, I) = \mathrm{H}_2(B, \mathbb{Z})$ (group homology).
- One consequence of current work: this also works for higher homology groups.
- Topological example: get usual fundamental group.

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Natural transformation κ

$$\kappa\colon \pi_1(-,I)\circ \mathrm{Cod} \Longrightarrow \mathrm{Gal}_\Gamma(-,0)$$

has components

$$\kappa_p\colon \pi_1(B,I) = \mathrm{Gal}_\Gamma(u,0) \longrightarrow \mathrm{Gal}_\Gamma(p,0)$$

for normal extension $p\colon E \longrightarrow B$, induced by (any)

$$\begin{tikzcd} U \arrow[r, dotted, "h"] \arrow[d, "u"'] & E \arrow[d, "p"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

That is, $\kappa_p = \mathrm{Gal}_\Gamma((h, 1_B), 0)$.

Universality of κ

Given

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define $\alpha\colon F \Longrightarrow \pi_1(-, I)$ by $\alpha_B = \gamma_u\colon FB \longrightarrow \pi_1(B, I)$.
Then

- α is natural by naturality of γ;
- $\kappa_p \circ \alpha_{\mathrm{Cod}\,p} = \gamma_p$ for all normal extensions p, by naturality of γ;
- α is unique: given β with $\kappa_p \circ \beta_{\mathrm{Cod}\,p} = \gamma_p$ for all normal p, get $\alpha_B = \beta_B$ as κ_u is an iso for weakly universal u.

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Kan extension

So indeed we have a Kan extension

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What we used

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "H"'] \arrow[dr, "G"] \arrow[d, phantom, "\kappa \Uparrow"] & \\ \mathscr{D} \arrow[rr, dotted, "K"'] & & \mathscr{B} \end{tikzcd}$$

- $H(f) = H(g) \Rightarrow G(f) = G(g)$
- for all $D \in \mathscr{D}$ there is $U \in \mathscr{A}$ with $H(U) = D$ and for all $A \in \mathscr{A}$, $\mathrm{Hom}_{\mathscr{A}}(U, A) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(D, HA)$ is surjective.
- Define $K(D) = G(U)$, well-defined and functorial by above properties.
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Next steps

Get also Kan extension

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\begin{tikzcd}[column sep=huge, row sep=huge]
& \mathrm{NExt}_\Gamma(\mathscr{C}) \arrow[dl, "\mathrm{Cod}"'] \arrow[dr, "U\circ \mathrm{Gal}_\Gamma(-{,}0)" description] \arrow[dr, bend left=60, "\mathrm{Ker}"] \arrow[d, phantom, "\kappa\Nearrow"] \arrow[dr, phantom, bend left=30, "\iota\Nearrow"] & \\
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- For $p\colon E \longrightarrow B$ normal, $U\circ \mathrm{Gal}_\Gamma(p,0) = \mathrm{Ker}\, p \cap \mathrm{Ker}\, \eta_E$.
- So $\iota_p\colon \mathrm{Ker}\, p \cap \mathrm{Ker}\, \eta_E \longrightarrow \mathrm{Ker}\, p$ is a mono.
- We hope that any natural transformation $F\circ \mathrm{Cod} \Longrightarrow \mathrm{Ker}$ factors over $U\circ \mathrm{Gal}_\Gamma(-,0)$. (true in the examples)
- In any case, can show another way that this is also a Kan extension.

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Thanks for listening!