Euler characteristics of colimits

Mike Shulman[1] Kate Ponto[2]

[1]University of California, San Diego
San Diego, California

[2]University of Kentucky
Lexington, Kentucky

PSSL 93
Cambridge, UK
April 14, 2012

Outline

1. Background
2. Symmetric monoidal traces
3. Bicategorical traces
4. Traces for enriched modules

The question

Let **A** be a finite category and $X\colon \mathbf{A} \to \mathbf{FinSet}$ a functor.

Question

When can the cardinality of $\operatorname{colim} X$ be calculated from cardinality information about X?

Example #1: Coproducts

For finite sets X and Y, we have

$$|X \sqcup Y| = |X| + |Y|$$

Example #2: Pushouts

For a pushout diagram of finite sets

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, "p"] \\
Z \arrow[r, "q"'] & Y \cup_X Z \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

if i and j are injections, then

$$|Y \cup_X Z| = |Y| + |Z| - |X|$$

Example #3: Quotients

For an action of a finite group G on a finite set X, if the action is free, then

$$\left|X/G\right| = \frac{|X|}{|G|}$$

Weightings

Definition

A weighting on a finite category **A** is a function

$$k^{(-)} \colon \mathrm{ob}(\mathbf{A}) \to \mathbb{Q}$$

such that . . .

Theorem (Leinster)

If **A** *admits a weighting and* $X \colon \mathbf{A} \to \mathbf{FinSet}$ *is a coproduct of representables, then*

$$\left|\operatorname{colim} X\right| = \sum_{a \in \mathbf{A}} k^a |X(a)|.$$

Weightings

Definition

A weighting on a finite category **A** is a function

$$k^{(-)}\colon \mathrm{ob}(\mathbf{A}) \to \mathbb{Q}$$

such that . . .

Theorem (Leinster)

If **A** *admits a weighting and* $X\colon \mathbf{A} \to \mathbf{FinSet}$ *is a coproduct of representables, then*

$$\Big|\operatorname{colim} X\Big| = \sum_{a\in\mathbf{A}} k^a |X(a)|.$$

What about more general diagrams?

Example #2: Homotopy pushouts

Regard a finite set as a 0-dimensional manifold.
Then its cardinality is equal to its Euler characteristic.

Theorem

For any homotopy pushout square of spaces with Euler characteristic:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & Y \cup_X Z \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

we have

$$\chi(Y \cup_X Z) = \chi(Y) + \chi(Z) - \chi(X)$$

Example #3: The lemma that is not Burnside's

Theorem (Cauchy, Frobenius)

For any action of a finite group G on a finite set X, we have

$$\left|X/G\right| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

where $X^g = \{\, x \in X \mid g \cdot x = x \,\}$.

If the action is free, then $X^e = X$ and $X^g = \emptyset$ for $g \neq e$.

Outline

1. Background
2. Symmetric monoidal traces
3. Bicategorical traces
4. Traces for enriched modules

Traces in symmetric monoidal categories

Definition

An object X of a closed symmetric monoidal category $\mathscr{V}$ is dualizable if we have X^* with maps

$$I \xrightarrow{\eta} X \otimes X^* \qquad X^* \otimes X \xrightarrow{\varepsilon} I$$

satisfying the triangle identities.

Traces in symmetric monoidal categories

Definition

An object X of a closed symmetric monoidal category $\mathscr{V}$ is dualizable if we have X^* with maps

$$I \xrightarrow{\eta} X \otimes X^* \qquad X^* \otimes X \xrightarrow{\varepsilon} I$$

satisfying the triangle identities.

Definition

If X is dualizable and $f\colon X \to X$, the trace of f is

$$I \xrightarrow{\eta} X \otimes X^* \xrightarrow{f \otimes 1} X \otimes X^* \xrightarrow{\cong} X^* \otimes X \xrightarrow{\varepsilon} I$$

The Euler characteristic of X is $\chi(X) = \mathrm{tr}(1_X)$.

Euler characteristics of finite sets

In $(\mathbf{FinSet}, \times, 1)$ not many objects are dualizable, but we can apply a monoidal functor

$$\Sigma\colon (\mathbf{FinSet}, \times, 1) \to (\mathscr{V}, \otimes, I)$$

which preserves some colimits, and calculate traces in $\mathscr{V}$.

Euler characteristics of finite sets

In $(\textbf{FinSet}, \times, 1)$ not many objects are dualizable, but we can apply a monoidal functor

$$\Sigma\colon (\textbf{FinSet}, \times, 1) \to (\mathscr{V}, \otimes, I)$$

which preserves some colimits, and calculate traces in $\mathscr{V}$.

Examples

- $\mathscr{V} = \textbf{Vect}$, $\Sigma X =$ the free vector space on X.

$$\chi(\Sigma X) = \dim(\Sigma X) = |X|.$$

Euler characteristics of finite sets

In $(\mathbf{FinSet}, \times, 1)$ not many objects are dualizable, but we can apply a monoidal functor

$$\Sigma \colon (\mathbf{FinSet}, \times, 1) \to (\mathscr{V}, \otimes, I)$$

which preserves some colimits, and calculate traces in $\mathscr{V}$.

Examples

- $\mathscr{V} = \mathbf{Vect}$, $\Sigma X =$ the free vector space on X.

$$\chi(\Sigma X) = \dim(\Sigma X) = |X|.$$

- $\mathscr{V} =$ the stable homotopy category, $\Sigma X =$ the suspension spectrum of X_+.

$$\chi(\Sigma X) = |X|.$$

Examples

1. If $\mathscr{V}$ is additive, then

$$\chi(X \oplus Y) = \chi(X) + \chi(Y)$$

2. (J.P. May) If $\mathscr{V}$ is triangulated, then

$$\chi(Y \cup_X Z) = \chi(Y) + \chi(Z) - \chi(X).$$

3. (Induced character) If $\mathscr{V}$ is additive and $|G|$-divisible, then

$$\chi(X/G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}_X(g).$$

(And similarly for traces of other endomorphisms.)

A more refined question

Suppose:

- $\mathscr{V}$ is closed symmetric monoidal and cocomplete.
- $\mathbf{A}$ is a small $\mathscr{V}$-category
- $\Phi \colon \mathbf{A}^{op} \to \mathscr{V}$ is a $\mathscr{V}$-functor (a "weight")
- $X \colon \mathbf{A} \to \mathscr{V}$ is a $\mathscr{V}$-functor with each $X(a)$ dualizable.

Questions

1. When does it follow that $\operatorname{colim}^{\Phi} X$ is dualizable?
2. Can we calculate $\chi(\operatorname{colim}^{\Phi} X)$ in terms of X?

A more refined question

Suppose:

- $\mathscr{V}$ is closed symmetric monoidal and cocomplete.
- $\mathbf{A}$ is a small $\mathscr{V}$-category
- $\Phi\colon \mathbf{A}^{op} \to \mathscr{V}$ is a $\mathscr{V}$-functor (a "weight")
- $X\colon \mathbf{A} \to \mathscr{V}$ is a $\mathscr{V}$-functor with each $X(a)$ dualizable.

Questions

1. When does it follow that $\operatorname{colim}^{\Phi} X$ is dualizable?
2. Can we calculate $\chi(\operatorname{colim}^{\Phi} X)$ in terms of X?

Remark: we allow $\mathscr{V}$ to have homotopy theory too: an "$(\infty, 1)$-category" or "derivator".

Outline

1. Background
2. Symmetric monoidal traces
3. Bicategorical traces
4. Traces for enriched modules

Duality in bicategories

- A monoidal $\mathscr{V}$ becomes a bicategory $B\mathscr{V}$ with one object.
- An object $X \in \mathscr{V}$ is dualizable $\iff$ X has an adjoint in $B\mathscr{V}$.

Question

In an arbitrary bicategory, given a 1-cell $X\colon A \nrightarrow B$ with an adjoint and a 2-cell $f\colon X \to X$, can we define its trace?

Duality in bicategories

- A monoidal $\mathscr{V}$ becomes a bicategory $B\mathscr{V}$ with one object.
- An object $X \in \mathscr{V}$ is dualizable $\iff$ X has an adjoint in $B\mathscr{V}$.

Question

In an arbitrary bicategory, given a 1-cell $X \colon A \nrightarrow B$ with an adjoint and a 2-cell $f \colon X \to X$, can we define its trace?

$$I_A \xrightarrow{\eta} X \odot X^* \xrightarrow{f \odot 1} X \odot X^* \xrightarrow{???} X^* \odot X \xrightarrow{\varepsilon} I_B$$

$X \odot X^*$ and $X^* \odot X$ don't even live in the same category!

Trace in bicategories

Suppose the bicategory is symmetric monoidal. If the object A has a dual, then a 1-cell $M\colon A \nrightarrow A$ has a trace:

$$\begin{tikzcd}
I \arrow[r, "\eta", "|" marking] \arrow[rrrr, bend right=15, "\mathrm{Tr}(M)"', "|" marking] & A \otimes A^* \arrow[r, "M\otimes I_{A^*}", "|" marking] & A \otimes A^* \arrow[r, "\cong", "|" marking] & A^* \otimes A \arrow[r, "\varepsilon", "|" marking] & I
\end{tikzcd}$$

Trace in bicategories

Suppose the bicategory is symmetric monoidal. If the object A has a dual, then a 1-cell $M\colon A \nrightarrow A$ has a trace:

$$\begin{tikzcd}
I \arrow[r, "\eta", "\shortmid" marking] \arrow[rrrr, bend right, "\mathrm{Tr}(M)"', "\shortmid" marking] & A \otimes A^* \arrow[r, "M \otimes I_{A^*}", "\shortmid" marking] & A \otimes A^* \arrow[r, "\cong", "\shortmid" marking] & A^* \otimes A \arrow[r, "\varepsilon", "\shortmid" marking] & I
\end{tikzcd}$$

Solution (Ponto)

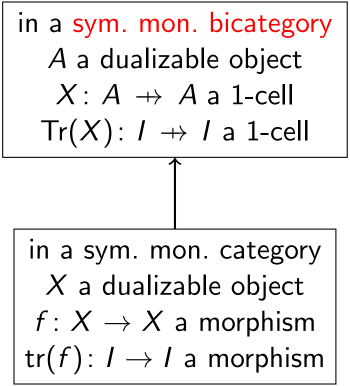
If the objects A and B have duals and $X\colon A \nrightarrow B$ has an adjoint, then $f\colon X \to X$ has a trace:

$$\begin{tikzcd}
\mathrm{Tr}(I_A) \arrow[r, "\eta"] \arrow[rrrr, bend right, "\mathrm{tr}(f)"'] & \mathrm{Tr}(X \odot X^*) \arrow[r, "f \odot 1"] & \mathrm{Tr}(X \odot X^*) \arrow[r, "\cong"] & \mathrm{Tr}(X^* \odot X) \arrow[r, "\varepsilon"] & \mathrm{Tr}(I_B)
\end{tikzcd}$$

A step back

in a sym. mon. category
X a dualizable object
$f\colon X \to X$ a morphism
$\mathrm{tr}(f)\colon I \to I$ a morphism

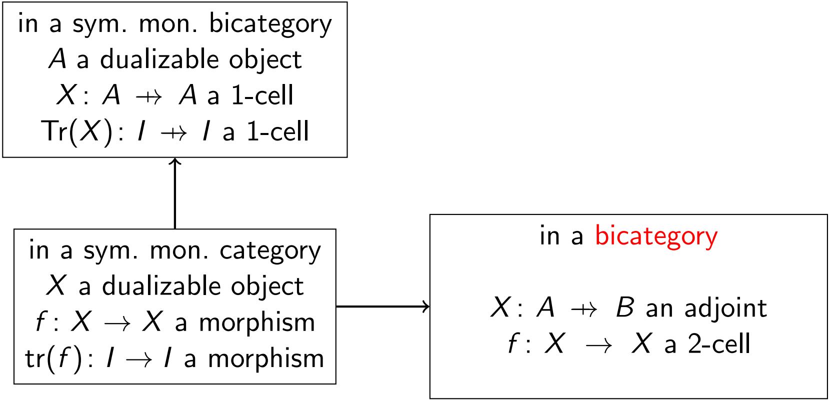
A step back



in a sym. mon. bicategory
A a dualizable object
$X\colon A \nrightarrow A$ a 1-cell
$\mathrm{Tr}(X)\colon I \nrightarrow I$ a 1-cell

in a sym. mon. category
X a dualizable object
$f\colon X \to X$ a morphism
$\mathrm{tr}(f)\colon I \to I$ a morphism

A step back



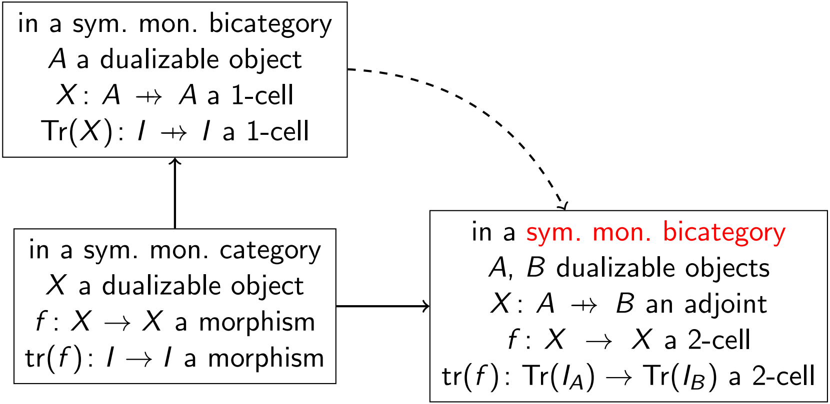
in a sym. mon. bicategory
A a dualizable object
$X\colon A \nrightarrow A$ a 1-cell
$\mathrm{Tr}(X)\colon I \nrightarrow I$ a 1-cell

in a sym. mon. category
X a dualizable object
$f\colon X \to X$ a morphism
$\mathrm{tr}(f)\colon I \to I$ a morphism

in a bicategory

$X\colon A \nrightarrow B$ an adjoint
$f\colon X \to X$ a 2-cell

A step back

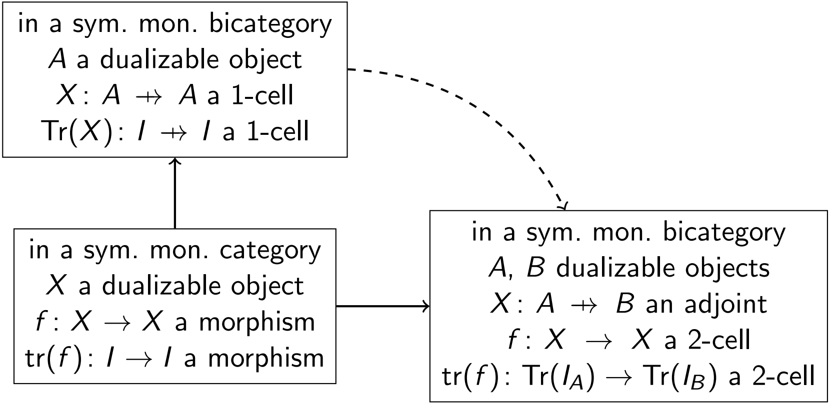


in a sym. mon. bicategory
A a dualizable object
$X\colon A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} A$ a 1-cell
$\mathrm{Tr}(X)\colon I \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} I$ a 1-cell

in a sym. mon. category
X a dualizable object
$f\colon X \to X$ a morphism
$\mathrm{tr}(f)\colon I \to I$ a morphism

in a sym. mon. bicategory
A, B dualizable objects
$X\colon A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ an adjoint
$f\colon X \to X$ a 2-cell
$\mathrm{tr}(f)\colon \mathrm{Tr}(I_A) \to \mathrm{Tr}(I_B)$ a 2-cell

A step back



in a sym. mon. bicategory
A a dualizable object
$X \colon A \nrightarrow A$ a 1-cell
$\mathrm{Tr}(X) \colon I \nrightarrow I$ a 1-cell

in a sym. mon. category
X a dualizable object
$f \colon X \to X$ a morphism
$\mathrm{tr}(f) \colon I \to I$ a morphism

in a sym. mon. bicategory
A, B dualizable objects
$X \colon A \nrightarrow B$ an adjoint
$f \colon X \to X$ a 2-cell
$\mathrm{tr}(f) \colon \mathrm{Tr}(I_A) \to \mathrm{Tr}(I_B)$ a 2-cell

(The Baez-Dolan microcosm principle.)

Composition of traces

If X and Y have adjoints, so does $X \odot Y$ (of course).

Theorem

For 2-cells $f : X \to X$ and $g : Y \to Y$, we have

$$\begin{tikzcd}
\mathrm{Tr}(I_A) \arrow[r, "\mathrm{tr}(f)"'] \arrow[rr, bend left, "\mathrm{tr}(f \odot g)"] & \mathrm{Tr}(I_B) \arrow[r, "\mathrm{tr}(g)"'] & \mathrm{Tr}(I_C)
\end{tikzcd}$$

Outline

1. Background
2. Symmetric monoidal traces
3. Bicategorical traces
4. Traces for enriched modules

The bicategory of enriched modules

Let $\mathscr{V}$ be symmetric monoidal closed and cocomplete.

Definition

The symmetric monoidal bicategory $\mathscr{V}$**Mod** has

- As objects, small $\mathscr{V}$-categories.
- As 1-cells $\mathbf{A} \nrightarrow \mathbf{B}$, $\mathscr{V}$-functors $\mathbf{B}^{op} \otimes \mathbf{A} \to \mathscr{V}$ (a.k.a. profunctors, distributors, modules, relators, . . .)
- The composite of $X\colon \mathbf{A} \nrightarrow \mathbf{B}$ and $Y\colon \mathbf{B} \nrightarrow \mathbf{C}$ is
$$(X \odot Y)(c,a) = \int^{b\in\mathbf{B}} X(b,a) \otimes Y(c,b)$$
- Every object $\mathbf{A}$ has a dual $\mathbf{A}^{op}$.
- The trace of $M\colon \mathbf{A} \nrightarrow \mathbf{A}$ is
$$\mathrm{Tr}(M) = \int^{a\in\mathbf{A}} M(a,a).$$

Diagrams and weights

Let $\mathbf{I}$ be the unit $\mathscr{V}$-category. Then

- A module $\mathbf{A} \nrightarrow \mathbf{I}$ is just a $\mathscr{V}$-functor $\mathbf{A} \to \mathscr{V}$ (a diagram).
- A module $\mathbf{I} \nrightarrow \mathbf{A}$ is just a $\mathscr{V}$-functor $\mathbf{A}^{op} \to \mathscr{V}$ (a weight).
- For $\Phi \colon \mathbf{I} \nrightarrow \mathbf{A}$ and $X \colon \mathbf{A} \nrightarrow \mathbf{I}$ we have

$$\operatorname{colim}^{\Phi} X \cong \Phi \odot X.$$

Dualizable modules

Theorem

A diagram X *has a right adjoint* $\iff$ *each* $X(a)$ *is dualizable.*

Dualizable modules

Theorem

A diagram X has a right adjoint $\iff$ each $X(a)$ is dualizable.

Theorem (Street)

A weight has a right adjoint $\iff$ it is absolute, i.e. Φ-weighted colimits are preserved by all $\mathscr{V}$-functors.

Absolute colimits are dualizable

Recall:

Question 1

If each $X(a)$ is dualizable, when is $\operatorname{colim}^{\Phi} X$ dualizable?

Answer

When Φ is absolute.

Absolute colimits are dualizable

Recall:

Question 1

If each $X(a)$ is dualizable, when is $\operatorname{colim}^{\Phi} X$ dualizable?

Answer

When Φ is absolute.

Examples

- Finite coproducts are absolute for additive $\mathscr{V}$.
- Pushouts are absolute for triangulated $\mathscr{V}$ (homotopically).
- Quotients by finite G are absolute for $|G|$-divisible $\mathscr{V}$.

Traces of absolute colimits

Question 2

If Φ is absolute and each $X(a)$ is dualizable, how can we calculate $\chi(\operatorname{colim}^{\Phi} X)$?

Abstract Answer

Since $\Phi\colon \mathbf{I} \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{A}$ and $X\colon \mathbf{A} \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{I}$ have adjoints, we have

$$\begin{tikzcd}
I = \mathrm{Tr}(I_{\mathbf{I}}) \arrow[r, "\mathrm{tr}(1_{\Phi})"'] \arrow[rr, bend left, "\chi(\operatorname{colim}^{\Phi} X) = \mathrm{tr}(1_{\Phi} \odot 1_X)"] & \mathrm{Tr}(I_{\mathbf{A}}) \arrow[r, "\mathrm{tr}(1_X)"'] & \mathrm{Tr}(I_{\mathbf{I}}) = I
\end{tikzcd}$$

But what are $\mathrm{Tr}(I_{\mathbf{A}})$, $\mathrm{tr}(1_{\Phi})$, and $\mathrm{tr}(1_X)$?

Traces of categories

$$\mathrm{Tr}(I_{\mathbf{A}}) = \int^{a\in\mathbf{A}} \mathbf{A}(a,a)$$

$$= \sum_{a\in\mathbf{A}} \mathbf{A}(a,a) \Big/ (\alpha\beta \sim \beta\alpha)$$

The coproduct (or “direct sum”) of all endomorphisms in **A**, modulo “conjugacy”.

Traces of categories

$$\mathrm{Tr}(I_{\mathbf{A}}) = \int^{a \in \mathbf{A}} \mathbf{A}(a,a)$$

$$= \sum_{a \in \mathbf{A}} \mathbf{A}(a,a) \Big/ (\alpha\beta \sim \beta\alpha)$$

The coproduct (or "direct sum") of all endomorphisms in **A**, modulo "conjugacy".

In particular, it contains

- a class for each identity morphism $[1_a]$.
- $[1_a] = [1_b]$ if and only if $a \cong b$.
- but also classes for other endomorphisms.

Traces for weights

For Φ an absolute weight, the trace of 1_Φ

$$\mathrm{tr}(1_\Phi)\colon I \to \mathrm{Tr}(I_{\mathbf{A}})$$

is a linear combination of conjugacy classes of endomorphisms:

$$\sum_\alpha \phi^\alpha[\alpha].$$

Theorem

If $\Phi = \Delta_{\mathbf{A}} 1$ is absolute, $\mathbf{A}$ is skeletal, and has no nonidentity endomorphisms, then $k^a := \phi^{1_a}$ defines a weighting on $\mathbf{A}$.

Traces for diagrams

Theorem

For X a dualizable diagram, the trace of 1_X

$$\operatorname{tr}(1_X)\colon \operatorname{Tr}(I_{\mathbf{A}}) \to I$$

sends each endomorphism $\alpha\colon a \to a$ in $\mathbf{A}$ to the trace in $\mathscr{V}$ of

$$X(a) \xrightarrow{X(\alpha)} X(a).$$

In particular, it sends 1_a to $\chi(X(a))$.

Traces of colimits

Recall:

$$\begin{tikzcd}
I = \mathrm{Tr}(I_{\mathbf{I}}) \arrow[r, "\mathrm{tr}(1_\Phi)"'] \arrow[rr, bend left, "\chi(\mathrm{colim}^{\mathbf{A}} X) = \mathrm{tr}(1_\Phi \odot 1_X)"] & \mathrm{Tr}(I_{\mathbf{A}}) \arrow[r, "\mathrm{tr}(1_X)"'] & \mathrm{Tr}(I_{\mathbf{I}}) = I
\end{tikzcd}$$

Concrete answer

If Φ is absolute and each $X(a)$ is dualizable, then

$$\chi(\mathrm{colim}^{\Phi} X) = \sum_{[\alpha]\in\mathrm{Tr}(I_{\mathbf{A}})} \phi^{\alpha} \cdot \mathrm{tr}\big(X(\alpha)\big)$$

Examples

1. Coproducts: $\mathbf{A}$ discrete with objects a and b.
   - If $\mathscr{V}$ is additive, $\Phi = \Delta_{\mathbf{A}}1$ is absolute.
   - $\mathrm{Tr}(I_{\mathbf{A}})$ generated by 1_a and 1_b.
   - $\phi^{1_a} = \phi^{1_b} = 1$.

2. Pushouts: $\mathbf{A}$ is $(b \leftarrow c \rightarrow a)$.
   - If $\mathscr{V}$ is stable/triangluated, $\Phi = \Delta_{\mathbf{A}}1$ is absolute.
   - $\mathrm{Tr}(I_{\mathbf{A}})$ generated by 1_a, 1_b, and 1_c.
   - $\phi^{1_a} = \phi^{1_b} = 1$ and $\phi^{1_c} = -1$.

3. Quotients: $\mathbf{A}$ is a finite group G.
   - If $\mathscr{V}$ is $|G|$-divisible, $\Phi = \Delta_{\mathbf{A}}1$ is absolute.
   - $\mathrm{Tr}(I_{\mathbf{A}})$ generated by conjugacy classes in G.
   - $\phi^C = \frac{|C|}{|G|}$

Another example: splitting idempotents

$\mathbf{A}$ the free-living idempotent e on an object x.

- $\Phi = \Delta_{\mathbf{A}} 1$ is absolute for any $\mathscr{V}$.
- $\mathrm{Tr}(I_{\mathbf{A}})$ generated by 1_x and e.
- $\phi^{1_x} = 0$ and $\phi^e = 1$.

The colimit of an idempotent $e\colon X \to X$ is a splitting of it, and

$$\chi(X/e) = \mathrm{tr}(e).$$

Euler characteristics of categories

Let $\mathbf{A}$ be a finite category with no nonidentity endomorphisms.

1. $\Phi = \Delta_{\mathbf{A}}1$ can be constructed from pushouts, hence is absolute for triangulated $\mathscr{V}$.
2. The trace of $1_{\Delta_{\mathbf{A}}1}$ is a weighting on $\mathbf{A}$.
3. The homotopy colimit of the constant diagram $X(a) = 1$ is the classifying space $|N\mathbf{A}|$.

Thus we can deduce:

Theorem (Leinster)

For $\mathbf{A}$ as above, we have

$$\chi(|N\mathbf{A}|) = \sum_{a \in A} k^a = \text{the “Euler characteristic of } \mathbf{A}\text{”}.$$